AN AXIOMATIC APPROACH TO GRADIENTS WITH APPLICATIONS TO DIRICHLET AND OBSTACLE PROBLEMS BEYOND FUNCTION SPACES

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Abstract. We develop a framework for studying variational problems in Banach spaces with respect to gradient relations, which encompasses many of the notions of generalized gradients that appear in the literature. We stress the fact that our approach is not dependent on function spaces and therefore applies equally well to functions on metric spaces as to operator algebras. In particular, we consider analogues of Dirichlet and obstacle problems, as well as first eigenvalue problems, and formulate conditions for the existence of solutions and their uniqueness. Moreover, we investigate to what extent a lattice structure may be introduced on (ordered) Banach spaces via a norm-minimizing variational problem. A multitude of examples is provided to illustrate the versatility of our approach.

1. Introduction

In the classical theory of partial differential equations, one explores the existence of solutions (and their regularity) by extending spaces of differentiable functions to include functions with only a weak notion of derivative. Introducing L^p-spaces and Sobolev spaces has the advantage that one may exploit the completeness of these spaces in order to find weak solutions of differential equations. In doing so, one is forced to work with equivalence classes of functions, rather than single functions, and the classical value of a function at a point is, for some purposes, simply not relevant anymore. Consequently, one tends to use Banach space techniques to reach the desired results. In particular, when extending the theory to functions on more general spaces, it becomes apparent that abstract methods are useful as classical techniques may not be applicable.

Consider the Dirichlet problem for harmonic functions, i.e. to find a harmonic function with given boundary values in a bounded domain Ω in $\mathbf{R}^n$. This problem can equivalently be reformulated as finding the minimizer of the energy integral

$$\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 \, dx, \tag{1.1}$$

over all sufficiently smooth functions with given boundary values. In this note, we aim to give an axiomatic approach to such problems starting from a quite general

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notion of gradient, assuming only a weak form of linearity. Many particular examples of gradients, such as weak gradients, upper gradients in metric spaces, Hajłasz gradients and algebraic derivations, fall into this class. We shall also consider gradients with no relation to derivatives (cf. Section 8), as well as examples which come from higher-order differential operators, such as the Laplacian and biLaplacian (cf. Section 8.3). It deserves to be pointed out that the framework we develop depends neither on function spaces nor on the commutativity of multiplication and, therefore, applies equally well to noncommutative settings, such as operator algebras.

We start by introducing an abstract notion of gradient relation and define a Sobolev space based on it. We show that, under minimal assumptions, this generalized Sobolev space is always a Banach space and that functions therein possess a unique minimal gradient. In Theorem 3.2 we formulate sufficient conditions for the existence of solutions to the Dirichlet problem with respect to this minimal gradient in analogy with (1.1). Furthermore, in Proposition 3.4 we give a condition for the solution to be unique.

In addition to the Dirichlet problem, we also consider the obstacle problem as well as the first eigenvalue problem (strictly speaking the existence of minimizers for the Rayleigh quotient, cf. Theorem 5.3). To solve the obstacle problem we reformulate it as a Dirichlet problem, and we can thus use the Dirichlet problem theory to solve the obstacle problem. Already here one can see the power of our abstract approach, as one can rarely consider obstacle problems as special cases of Dirichlet problems in more traditional situations (cf. Remark 3.5). A prominent role in the minimization problems above is played by *Poincaré sets*, i.e. subsets $\mathcal{K}$ of the abstract Sobolev space which support a generalized Poincaré inequality:

$$\|u\| \leq C\,\|\nabla u\|, \quad u \in \mathcal{K}.$$

Such sets provide natural domains when considering variational problems in the context of gradient relations.

Finally, inspired by the fact that the pointwise maximum of two functions minimizes the L^p-norm among all functions which majorize both functions, we investigate the possibility of defining the maximum (as well as the minimum) of two elements in a Banach space via a minimization problem (cf. Propositions 6.4 and 6.12). Furthermore, we formulate necessary and sufficient conditions for the existence of least upper (resp. greatest lower) bounds (cf. Theorem 6.15).

The paper is organized as follows. Section 2 introduces the generalized concept of gradient that we shall be studying, as well as the corresponding concept of gradient space and the associated Sobolev space. These objects are the basic ingredients of our analysis. Sections 3 and 4 introduce the Dirichlet and obstacle problems together with the concept of preordered gradient spaces. It is shown that, under certain conditions, solutions of the Dirichlet and obstacle problems exist. In Section 5 we consider the Rayleigh quotient which, in classical analysis, is related to finding the first positive eigenvalue of the Laplace operator. Also here, a minimizer can be found under certain assumptions.

In Section 6 we investigate the possibility of defining a least upper bound via a minimization problem. While the least upper bound will in general not exist, we show that a minimizer (in norm) exists, retaining some of the properties of a least upper bound. Sections 7 and 8 are devoted to showing that many situations can be treated in a unified way within our framework. The examples include both classical function spaces and functions on metric spaces (together with their appropriate

concepts of gradients) as well as noncommutative examples such as spaces of matrix algebras, operator algebras and operator-valued functions.

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2. Gradient spaces

When moving away from the realm of differentiable functions defined on vector spaces, one is lead to introduce several (in general, different) concepts of a derivative, or gradient. For instance, one may consider weak derivatives on $\mathbf{R}^n$ or upper gradients on metric spaces. In this section, we will introduce a very weak abstract notion of gradient, which encompasses many of the situations one would like to study. There is actually no mathematical reason for using the name gradient (as the assumptions only include a weak form of linearity), but we have chosen to keep the terminology both for historical reasons and in view of many applications. Moreover, we shall introduce a corresponding pair of Banach spaces, which provides a link between the gradient and the analytic structure of the normed space.

Let us start by introducing the concept of a gradient, in the form of a relation on the Cartesian product of two vector spaces.

Definition 2.1. Let $\widetilde{V}$ and $\widetilde{W}$ be vector spaces over $\mathbf{R}$ or $\mathbf{C}$ (not necessarily the same for $\widetilde{V}$ and $\widetilde{W}$). A *gradient relation* is a relation $R \subseteq \widetilde{V} \times \widetilde{W}$ such that

(G1) if $(u, g) \in R$ and $(u', g') \in R$ then $(u + u', g + g') \in R$,
(G2) if $(u, g) \in R$ and $\alpha > 0$ then $(\alpha u, \alpha g) \in R$.

We say that g *is a gradient of* u if $(u, g) \in R$.

To be able to address analytic questions, we introduce subspaces V and W, of $\widetilde{V}$ and $\widetilde{W}$, which have the structure of Banach spaces. Note that property (GS4) below is important because it connects the gradient relation to the analytic structure of the space.

Definition 2.2. A *gradient space* $\mathcal{U} = \big(V, \widetilde{V}, W, \widetilde{W}, R\big)$ consists of two vector spaces $\widetilde{V}$ and $\widetilde{W}$ together with a gradient relation $R \subseteq \widetilde{V} \times \widetilde{W}$ and linear subspaces $V \subseteq \widetilde{V}$ and $W \subseteq \widetilde{W}$ such that

(GS1) V is a reflexive Banach space,
(GS2) W is a reflexive and strictly convex Banach space,
(GS3) if $(u, g) \in R$ with $u \in V$ and $g \in W$, then there exists $g' \in W$ such that $(-u, g') \in R$,
(GS4) if $u, u_i \in V$ and $g, g_i \in W$ with $(u_i, g_i) \in R$, for $i = 1, 2, \ldots$, are such that $\|u - u_i\|_V \to 0$ and $\|g - g_i\|_W \to 0$, then $(u, g) \in R$,

where $\|\cdot\|_V$ and $\|\cdot\|_W$ denote the norms of V and W, respectively.

Note that for classical derivatives and gradients, as well as for weak gradients, one has $g' = -g$ in (GS3), while for (weak) upper gradients and Hajłasz gradients (see Sections 7.3–7.5) one has $g' = g$. In (GS3) we allow for even more general situations, and do not even require a uniform bound of the form $\|g'\|_W \le C \|g\|_W$.

Just as for L^p-spaces, it is natural to introduce a subspace of V, consisting of elements which have a gradient in W.

Assume for the rest of this section that $\mathcal{U} = (V, \widetilde{V}, W, \widetilde{W}, R)$ *is a gradient space.*

Definition 2.3. The set

$$\mathrm{Sob}(\mathcal{U}) := \{u \in V : (u,g) \in R \text{ for some } g \in W\}$$

is called the *Sobolev space of the gradient space* $\mathcal{U}$.

Lemma 2.4. *If* $\mathrm{Sob}(\mathcal{U}) \neq \varnothing$*, then* $\mathrm{Sob}(\mathcal{U})$ *is a vector space over* $\mathbf{R}$*.*

Proof. As $\mathrm{Sob}(\mathcal{U}) \neq \varnothing$, there exists $u \in \mathrm{Sob}(\mathcal{U})$ which, by (GS3), implies that $-u \in \mathrm{Sob}(\mathcal{U})$. From (G1) it follows that if $u, u' \in \mathrm{Sob}(\mathcal{U})$ then $u+u' \in \mathrm{Sob}(\mathcal{U})$, since V and W are vector spaces. In particular, this implies that $u+(-u) = 0 \in \mathrm{Sob}(\mathcal{U})$. Finally, from (G2) and (GS3) (and the fact that $0 \in \mathrm{Sob}(\mathcal{U})$) it follows that if $u \in \mathrm{Sob}(\mathcal{U})$ and $\alpha \in \mathbf{R}$ then $\alpha u \in \mathrm{Sob}(\mathcal{U})$. $\square$

Lemma 2.5. *If* $\mathrm{Sob}(\mathcal{U}) \neq \varnothing$*, then* $(0,0) \in R$*.*

Proof. If $\mathrm{Sob}(\mathcal{U}) \neq \varnothing$, then $0 \in \mathrm{Sob}(\mathcal{U})$, by Lemma 2.4, and thus there is $g \in W$ such that $(0,g) \in R$. Hence, $(0, g/k) \in R$ for $k = 1,2,\ldots$, by (G2). Since $\|g/k\|_W = \|g\|_W/k \to 0$, as $k \to \infty$, it follows from (GS4) that $(0,0) \in R$. $\square$

The following results (Lemma 2.6, Corollary 2.7 and Lemma 2.8) are fairly standard, but we have chosen to repeat them here adjusted to our setting. They constitute a set of very useful technical results, and concern the possibility of constructing strongly convergent sequences from weakly convergent sequences, by using convex combinations.

Lemma 2.6 (Theorem 3.12 in Rudin [17])**.** *If E is a convex subset of a locally convex space, then the weak closure of E equals the* (*strong*) *closure of E.*

The following results can be found in the literature under the name Mazur's lemma, with the slight difference that the linear combinations are usually taken starting from $j = 1$. To prove Mazur's lemma in the form given here, an iterative argument is then needed. We therefore provide a full proof of the result, tailored to our needs. Note that the sums below start at $j = i$, which will be important when we apply this result. Recall that a *convex combination* is a linear combination with nonnegative coefficients summing up to one, such that only finitely many of them are positive.

Corollary 2.7. *Let $\{u_i\}_{i=1}^\infty$ be a weakly convergent sequence with weak limit u in a normed vector space. Then there exist convex combinations*

$$\tilde{u}_i = \sum_{j=i}^{N_i} \alpha_{ij} u_j \quad \text{with } \alpha_{ij} \geq 0 \text{ and } \sum_{j=i}^{N_i} \alpha_{ij} = 1,$$

such that the sequence $\{\tilde{u}_i\}_{i=1}^\infty$ is strongly convergent to u.

Proof. Let E_i denote the convex hull of $\{u_i, u_{i+1}, \ldots\}$ (i.e. the set of its convex combinations), and let $\overline{E}_i^w$ and $\overline{E}_i$ denote the weak and strong closures of E_i, respectively. By assumption, $u \in \overline{E}_i^w$ for $i = 1,2,\ldots$. Furthermore, it follows from Lemma 2.6 that $u \in \overline{E}_i$ for $i = 1,2,\ldots$, which implies that there exists, for each i, $\tilde{u}_i \in E_i$ such that

$$\|\tilde{u}_i - u\| < \frac{1}{i}.$$

Thus $\tilde{u}_i \to u$ strongly, and as $\tilde{u}_i \in E_i$, it is a convex combination of the elements $u_i, u_{i+1}, u_{i+2}, \ldots$. $\square$

Lemma 2.8. *Let $\{u_i\}_{i=1}^{\infty} \subseteq V$ and $\{g_i\}_{i=1}^{\infty} \subseteq W$ be bounded sequences such that $(u_i, g_i) \in R$ for $i = 1, 2, \ldots$. Then there exist convex combinations*

$$\tilde{u}_i = \sum_{j=i}^{N_i} \alpha_{ij} u_j \quad \text{and} \quad \tilde{g}_i = \sum_{j=i}^{N_i} \alpha_{ij} g_j,$$

with limits $u := \lim_{i\to\infty} \tilde{u}_i$ and $g := \lim_{i\to\infty} \tilde{g}_i$, such that $(u, g) \in R$.

Proof. Since $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence and V is reflexive (by property (GS1)), Banach–Alaoglu's theorem shows that there exists a weakly convergent subsequence, with some weak limit u, which, by a slight abuse of notation, we shall also denote by $\{u_i\}_{i=1}^{\infty}$. Moreover, we assume that $\{g_i\}_{i=1}^{\infty}$ denotes the corresponding (not necessarily weakly convergent) subsequence of gradients. From Corollary 2.7 it follows that there exist convex combinations

$$\hat{u}_i = \sum_{j=i}^{\widehat{N}_i} \widehat{\alpha}_{ij} u_j$$

such that the sequence $\{\hat{u}_i\}_{i=1}^{\infty}$ converges (strongly) to u. The corresponding linear combinations of gradients

$$\hat{g}_i = \sum_{j=i}^{\widehat{N}_i} \widehat{\alpha}_{ij} g_j$$

fulfill $(\hat{u}_i, \hat{g}_i) \in R$ for $i = 1, 2, \ldots$ (which follows from (G1) and (G2)). Now, since $\{\hat{g}_i\}_{i=1}^{\infty}$ is still a bounded sequence and W is reflexive, Banach–Alaoglu's theorem again shows that there exists a weakly convergent subsequence $\{\hat{g}_{j_k}\}_{k=1}^{\infty}$ with some weak limit g. Using Corollary 2.7 once more, one finds convex combinations

$$\tilde{g}_i = \sum_{k=i}^{N_i} \widetilde{\alpha}_{ik} \hat{g}_{j_k}$$

such that the sequence $\{\tilde{g}_i\}_{i=1}^{\infty}$ converges (strongly) to g. The corresponding linear combinations

$$\tilde{u}_i = \sum_{k=i}^{N_i} \widetilde{\alpha}_{ik} \hat{u}_{j_k}$$

fulfill $(\tilde{u}_i, \tilde{g}_i) \in R$ for $i = 1, 2, \ldots$. Finally, let us show that $\{\tilde{u}_i\}_{i=1}^{\infty}$ converges to u. Namely, since the sequence $\{\hat{u}_i\}_{i=1}^{\infty}$ converges to u, the subsequence $\{\hat{u}_{j_k}\}_{k=1}^{\infty}$ converges to u. Furthermore,

$$\|\tilde{u}_i - u\|_V = \left\| \sum_{k=i}^{N_i} \widetilde{\alpha}_{ik} \hat{u}_{j_k} - \sum_{k=i}^{N_i} \widetilde{\alpha}_{ik} u \right\| \le \sum_{k=i}^{N_i} \widetilde{\alpha}_{ik} \left\|\hat{u}_{j_k} - u\right\|_V,$$

which shows that $\lim_{i\to\infty} \tilde{u}_i = u$ since $\lim_{i\to\infty} \hat{u}_{j_k} = u$. Hence, we have found two (strongly) convergent sequences $\{\tilde{u}_i\}_{i=1}^{\infty}$ and $\{\tilde{g}_i\}_{i=1}^{\infty}$ (given as convex combinations of the original bounded sequences), with $\lim_{i\to\infty} \tilde{u}_i = u$ and $\lim_{i\to\infty} \tilde{g}_i = g$, such that $(\tilde{u}_i, \tilde{g}_i) \in R$. By property (GS4) it follows that $(u, g) \in R$. ∎

For the Sobolev space $\mathrm{Sob}(\mathcal{U})$, it is natural to introduce the norm

$$\|u\|_{\mathrm{Sob}(\mathcal{U})} = \|u\|_V + \inf_{(u,g)\in R} \|g\|_W,$$

where the infimum is taken over all gradients g of u. Since $g+g'$ is a gradient of $u+u'$ (where g and g' are gradients of u and u', respectively) it is clear that the triangle inequality is fulfilled, making $\|\cdot\|_{\mathrm{Sob}(\mathcal{U})}$ a norm on $\mathrm{Sob}(\mathcal{U})$.

Theorem 2.9. *The Sobolev space* $\mathrm{Sob}(\mathcal{U})$ *is a Banach space.*

Proof. It is easy to check that $(\mathrm{Sob}(\mathcal{U}), \|\cdot\|_{\mathrm{Sob}(\mathcal{U})})$ is a normed space, so the only thing remaining is to show that $\mathrm{Sob}(\mathcal{U})$ is complete with respect to the given norm. Let $\{u_j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathrm{Sob}(\mathcal{U})$, i.e. for every $n=0,1,\ldots$, there exists an index k_n such that

$$\|u_j-u_k\|_{\mathrm{Sob}(\mathcal{U})} < 2^{-n} \quad \text{whenever } j,k \ge k_n.$$

In particular, $\|u_j-u_k\|_V < 2^{-n}$ and there exists a gradient g_{jk} of u_j-u_k such that $\|g_{jk}\|_W < 2^{-n}$ whenever $j,k \ge k_n$. It follows directly that $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in V, and thus has a limit $u := \lim_{j\to\infty} u_j$, as V is complete.

Our next aim is to show that $u_j \to u$ also in $\mathrm{Sob}(\mathcal{U})$, and we shall proceed by constructing gradients for $u_{k_l}-u$ that tend to zero in W as $l\to\infty$. Thus, for every j and l let

$$s_{lj} := u_{k_l} - u_{k_{l+j}} = \sum_{i=1}^{j} (u_{k_{l+i-1}} - u_{k_{l+i}}) \quad \text{and} \quad h_{lj} = \sum_{i=1}^{j} g_{k_{l+i-1},k_{l+i}},$$

from which it follows that h_{lj} is a gradient of s_{lj}. Moreover, we set

$$s_l := \lim_{j\to\infty} s_{lj} = u_{k_l} - u.$$

As

$$\|s_{lj}\|_V \le \sum_{i=1}^{j} 2^{-(l+i-1)} < 2^{1-l} \quad \text{and} \quad \|h_{lj}\|_W \le \sum_{i=1}^{j} 2^{-(l+i-1)} < 2^{1-l},$$

the sequences $\{s_{lj}\}_{j=1}^\infty$ and $\{h_{lj}\}_{j=1}^\infty$ are bounded. An application of Lemma 2.8 provides us with convex combinations

$$\tilde{s}_{li} = \sum_{j=i}^{N_i} \alpha_{ij} s_{lj} \quad \text{and} \quad \tilde{h}_{li} = \sum_{j=i}^{N_i} \alpha_{ij} h_{lj}$$

such that $\tilde{s}_{li} \to s_l$ in V and $\tilde{h}_{li} \to h_l$ in W, as $i\to\infty$, and $(s_l,h_l)\in R$. Furthermore,

$$\|h_l\|_W = \lim_{i\to\infty} \|\tilde{h}_{li}\|_W \le \varliminf_{i\to\infty} \|h_{li}\|_W < 2^{1-l}.$$

Finally, for every $k \ge k_l$, we see that $h_l + g_{k_l,k}$ is a gradient of $u_k - u = (u_k - u_{k_l}) + s_l$ and

$$\|h_l + g_{k_l,k}\|_W \le \|h_l\|_W + \|g_{k_l,k}\|_W < 2^{1-l} + 2^{-l} \to 0, \quad \text{as } l\to\infty.$$

Letting $l\to\infty$ we conclude that

$$\|u_k - u\|_{\mathrm{Sob}(\mathcal{U})} \le \|u_k-u\|_V + \|h_l + g_{k_l,k}\|_W \to 0, \quad \text{as } k\to\infty. \qquad \square$$

In general, there can be many gradients of an element $u\in V$, see e.g. Sections 7.3–7.6, but we shall mainly be interested in the minimal one (in the following sense).

Definition 2.10. Let $u\in V$. An element $g_u \in W$ is a *minimal gradient of* u if $(u,g_u)\in R$ and

$$\|g_u\|_W \le \|g\|_W \quad \text{for all } g\in W \text{ such that } (u,g)\in R.$$

Theorem 2.11. *Every element $u \in \mathrm{Sob}(\mathcal{U})$ has a unique minimal gradient.*

We will denote the minimal gradient of u by g_u. Note that, if $\alpha \geq 0$, then $g_{\alpha u} = \alpha g_u$ since αg_u is a gradient of $g_{\alpha u}$ and (when $\alpha > 0$)

$$\|g_u\|_W = \|g_{\alpha^{-1}\alpha u}\|_W \leq \alpha^{-1} \|g_{\alpha u}\|_W \leq \alpha^{-1}\alpha \|g_u\|_W = \|g_u\|_W ,$$

and the uniqueness in Theorem 2.11 shows that $g_{\alpha u} = \alpha g_u$.

Proof. Let $I = \inf_g \|g\|_W$, where the infimum is over all gradients of u in W, and let $\{g_j\}_{j=1}^\infty$ be a minimizing sequence, i.e.

$$\lim_{j\to\infty} \|g_j\|_W = I,$$

where g_j is a gradient of u for $j = 1, 2, \ldots$. The minimizing sequence is clearly bounded, and from Lemma 2.8 (with $u_i = u$ for $i = 1, 2, \ldots$) it follows that there exist convex combinations $\tilde{g}_i = \sum_{j=i}^{N_i} \alpha_{ij} g_j$ converging to some $g \in W$ with $(u, g) \in R$. Since g is a gradient of u one has $I \leq \|g\|_W$, and thus

$$I \leq \|g\|_W = \lim_{i\to\infty} \|\tilde{g}_i\|_W \leq \limsup_{j\to\infty} \|g_j\|_W = I,$$

which implies that $\|g\|_W = I$.

Let us now prove uniqueness. If g_1 and g_2 are two minimal gradients of u then $h = \frac{1}{2}(g_1 + g_2)$ is also a gradient of u, which implies that

$$I \leq \|h\|_W \leq \tfrac{1}{2}(\|g_1\|_W + \|g_2\|_W) = I$$

and we conclude that h is also a minimal gradient of u. Hence

$$\|g_1\|_W = \|g_2\|_W = \left\|\tfrac{1}{2}(g_1 + g_2)\right\|_W$$

and since W is assumed to be a strictly convex space, it follows that $\|g_1 - g_2\|_W = 0$, which proves that the minimal gradient is unique. □

3. The Dirichlet problem

Assume in this section that $\mathcal{U} = (V, \widetilde{V}, W, \widetilde{W}, R)$ is a gradient space.

The classical Dirichlet problem for harmonic (or p-harmonic) functions can be formulated as follows: subject to given boundary conditions in a domain, one tries to find a weakly differentiable function whose (weak) gradient has minimal norm among all such functions satisfying the boundary conditions.

In the setting of gradient spaces, we shall formulate the problem in the following way. Let $\mathcal{K}_0$ be a subset of $\mathrm{Sob}(\mathcal{U})$ (which, in the classical setting, corresponds to the set of functions in a domain with zero boundary values). Given $f \in \mathrm{Sob}(\mathcal{U})$ we set

$$\mathcal{K}_f = \mathcal{K}_0 + f = \{v \in \mathrm{Sob}(\mathcal{U}) : v - f \in \mathcal{K}_0\},$$

and think of $\mathcal{K}_f$ as the analogue of the set of functions which are equal to f on the boundary of the domain. A solution of the Dirichlet problem with respect to $\mathcal{K}_f$ is then given by an element $u \in \mathcal{K}_f$ such that

$$\|g_u\|_W = \inf_{v\in\mathcal{K}_f} \|g_v\|_W ,$$

where g_u and g_v denote the minimal gradients of u and v, respectively. As stated, the Dirichlet problem does not have enough analytic structure to ensure the existence of a solution in the general case. Therefore, a Poincaré inequality is introduced, which allows one to obtain a bound on the norm of an element in $\mathcal{K}_0$ in terms of the norm of its minimal gradient.

Definition 3.1. A *Poincaré set* is a subset $A \subseteq \operatorname{Sob}(\mathcal{U})$ for which there exists a constant $C > 0$ such that for all $u \in A$,

$$\|u\|_V \leq C \|g\|_W$$

for all $g \in W$ such that $(u, g) \in R$.

Under the assumption that $\mathcal{K}_0$ is a (closed and convex) Poincaré set one can obtain the existence of a solution to the Dirichlet problem. However, in general the solution will not be unique, see Examples 3.6 and 3.7.

Theorem 3.2. *For any nonempty closed convex Poincaré set $\mathcal{K}_0 \subseteq \operatorname{Sob}(\mathcal{U})$ and $f \in \operatorname{Sob}(\mathcal{U})$, the Dirichlet problem with respect to $\mathcal{K}_f = \mathcal{K}_0 + f$ has at least one solution. Moreover, if u_1 and u_2 are solutions to the Dirichlet problem, then $g_{u_1} = g_{u_2}$.*

Remark 3.3. The proof shows that if the minimizing sequence in the Dirichlet problem can a priori be assumed to be bounded in V, then the assumption that $\mathcal{K}_0$ is a Poincaré set can be omitted in Theorem 3.2. This observation was e.g. used in Björn–Björn [2, Theorem 5.13] to prove the existence of capacitary minimizers for arbitrary bounded condensers. See Section 5 therein for examples and further discussion on the role of Poincaré inequalities in the Dirichlet problem.

Proof. Let $\{u_j\}_{j=1}^\infty \subseteq \mathcal{K}_f$ be a minimizing sequence, i.e.

$$\lim_{j\to\infty} \|g_j\|_W = \inf_{v\in\mathcal{K}_f} \|g_v\|_W =: I,$$

where g_j denotes the minimal gradient of u_j. Clearly, the sequence $\{g_j\}_{j=1}^\infty$ is bounded. Since $\mathcal{K}_0$ is a Poincaré set there exists a $C > 0$ such that

$$\|u_j - f\|_V \leq C \left\|g_{u_j - f}\right\|_W,$$

and, by using (GS3), there exists a $g' \in W$ such that

$$\|u_j - f\|_V \leq C \left\|g_{u_j - f}\right\|_W \leq C'(\|g_j\|_W + \|g'\|_W),$$

which implies that $\{u_j\}_{j=1}^\infty$ is also a bounded sequence. From Lemma 2.8 it follows that there exist convex combinations $\{\tilde{u}_i\}_{i=1}^\infty$ and $\{\tilde{g}_i\}_{i=1}^\infty$, converging to some functions u and g, respectively, with $(u, g) \in R$. As $\mathcal{K}_0$ is convex we have $\tilde{u}_i \in \mathcal{K}_f$ and, furthermore, since $\mathcal{K}_0$ is closed, it follows that $u \in \mathcal{K}_f$. It remains to show that g_u is indeed a minimizer, which is easily seen from

$$I \leq \|g_u\|_W \leq \|g\|_W = \lim_{i\to\infty} \|\tilde{g}_i\|_W \leq I,$$

as $\{g_i\}_{i=1}^\infty$ is assumed to be a minimizing sequence (and $\tilde{g}_i$, given by Lemma 2.8, are convex combinations of g_i, $g_{i+1}, \ldots$).

Finally, let us prove that if u_1 and u_2 are two minimizers, then $g_{u_1} = g_{u_2}$. Since $\mathcal{K}_0$ is a convex set, the element $u = \frac{1}{2}(u_1 + u_2)$ is in $\mathcal{K}_f$, and it has a (not necessarily

minimal) gradient $\frac{1}{2}(g_{u_1}+g_{u_2})$. As $u \in \mathcal{K}_f$ we must have $\left\|\frac{1}{2}(g_{u_1}+g_{u_2})\right\|_W \geq I$. But from the triangle inequality one obtains

$$I \leq \left\|\tfrac{1}{2}(g_{u_1}+g_{u_2})\right\|_W \leq \tfrac{1}{2}\left\|g_{u_1}\right\|_W + \tfrac{1}{2}\left\|g_{u_2}\right\|_W = I,$$

which implies that $\left\|\frac{1}{2}(g_{u_1}+g_{u_2})\right\|_W = \left\|g_{u_1}\right\|_W = \left\|g_{u_2}\right\|_W$. Since W is strictly convex, it follows that $g_{u_1} = g_{u_2}$. □

There is one important special case, where the uniqueness of solutions can be assured. Namely, when the assignment $u \mapsto g_u$ is linear and $\mathcal{K}_0$ is a linear subspace (and not only a convex set).

Proposition 3.4. *If the map $u \mapsto g_u$ is linear and $\mathcal{K}_0 \subseteq \mathrm{Sob}(\mathcal{U})$ is a nonempty closed linear subspace, which is also a Poincaré set, then the Dirichlet problem with respect to $\mathcal{K}_f$ has a unique solution.*

Proof. By Theorem 3.2, the Dirichlet problem with respect to $\mathcal{K}_f$ has at least one solution. Let u_1 and u_2 be two solutions. By Theorem 3.2 we know that $g_{u_1} = g_{u_2}$. By assumption, the map $u \mapsto g_u$ is linear, from which it follows that $g_{u_1-u_2} = 0$. As $\mathcal{K}_0$ is a linear subspace, we have

$$u_1 - u_2 = (u_1 - f) - (u_2 - f) \in \mathcal{K}_0. \tag{3.1}$$

Now, since $\mathcal{K}_0$ is a Poincaré set, it follows that

$$\left\|u_1 - u_2\right\|_V \leq C \left\|g_{u_1-u_2}\right\|_W = 0,$$

which implies that $u_1 = u_2$. □

Remark 3.5. In the classical situation, $\mathcal{K}_0$ is usually the zero Sobolev space, and thus a closed linear subspace. Here (but for Proposition 3.4) we have merely assumed $\mathcal{K}_0$ to be a closed convex set (with no extra cost, as this is enough for the proofs). An advantage of this approach is that we can formulate the obstacle problem in the next section as a Dirichlet problem. In the obstacle problem the class of competing functions is only convex also in the classical situation, which traditionally means that it has to be handled separately. Alternatively one can treat the Dirichlet problem as a special case of the obstacle problem, but the opposite direction (to treat the obstacle problem as a Dirichlet problem) is not possible with traditional formulations of the problems.

Example 3.6. Let $V = \mathbf{C}$, $W = \mathbf{R}$ and define

$$R = \{(u,g) \in V \times W : g \geq \max(|\mathrm{Re}\, u|, |\mathrm{Im}\, u|)\} \quad \text{and} \quad \mathcal{K}_0 = \{u \in V : \mathrm{Re}\, u \geq 0\}. \tag{3.2}$$

Then $\mathcal{U} = (V, V, W, W, R)$ is a gradient space (note that we need an inequality, rather than equality, when defining R, for property (G1) to hold). Moreover, $\mathcal{K}_0$ is a closed convex Poincaré set, and $1 + ai$ is a solution of the Dirichlet problem with respect to $\mathcal{K}_f$, with $f = 1$, for all $|a| \leq 1$. Thus we do not always have uniqueness in Theorem 3.2. (In Examples 3.6 and 3.7, i denotes the imaginary unit.)

Example 3.7. Let $V = W^{1,2}(\mathbf{R}^2, \mathbf{C})$ (the space of complex-valued $W^{1,2}$ functions on $\mathbf{R}^2$, cf. Section 7.1), $W = L^2(\mathbf{R}^2, \mathbf{R})$ and define

$$R = \{(u,g) \in V \times W : g \geq \max(|\nabla\, \mathrm{Re}\, u|, |\nabla\, \mathrm{Im}\, u|) \text{ a.e. in } \mathbf{R}^2\},$$

where ∇v is the distributional gradient of v. It is easy to verify that $\mathcal{U} = (V, V, W, W, R)$ is a gradient space (again we need an inequality when defining R), and that

$\operatorname{Sob}(\mathcal{U}) = W^{1,2}(\mathbf{R}^2, \mathbf{C})$ (with an unorthodox gradient structure similar to the one in (3.2) and (7.3)).

Let $\Omega = \{x \in \mathbf{R}^2 : 1 < |x| < 2\}$, $\mathcal{K}_0 = W_0^{1,2}(\Omega, \mathbf{C})$ be the usual zero Sobolev space (which is a closed convex Poincaré set), and $f \in \operatorname{Sob}(\mathcal{U})$ be such that $f(x) = 1$ for $|x| \le 1$ and $f(x) = 0$ for $|x| \ge 2$. By Theorem 3.2, the Dirichlet problem with respect to $\mathcal{K}_f$ has a solution $v \in \operatorname{Sob}(\mathcal{U})$. Then $u_0 := \operatorname{Re} v$ is a real-valued solution of the same Dirichlet problem, since every gradient of v is also a gradient of u_0. In fact, $u_0(x) = 1 - \log_2 |x|$ is the unique solution of the classical Dirichlet problem for harmonic functions with the boundary data f. Let $\varphi : [0,1] \to [0,1]$ be a continuously differentiable function such that $\varphi(0) = \varphi(1) = 0$ and $|\varphi'(t)| \le 1$ for $0 \le t \le 1$. Also let $u(x) = u_0(x) + i\varphi(u_0(x))$. Then the minimal gradient of u is

$$g_u(x) = \max(|\nabla u_0(x)|, |\varphi'(u_0(x))|\,|\nabla u_0(x)|) = |\nabla u_0(x)|,$$

and u is also a solution of the Dirichlet problem with respect to $\mathcal{K}_f$, which hence does not have a unique solution, even though the existence of a solution is guaranteed by Theorem 3.2.

4. The obstacle problem

Let us now approach the obstacle problem for gradient spaces. That is, we would like to solve the Dirichlet problem given the extra constraint that the solution has to be larger than a given function (the obstacle). So far, there is no concept of ordering in a gradient space; hence, in the following we must assume that one may compare elements of V. For this purpose let us recall the definition of a linear preorder on a vector space.

Definition 4.1. Let $\widetilde{V}$ be a vector space. A *linear preorder* on $\widetilde{V}$ is a binary relation $\le$ such that for $a, b, c \in \widetilde{V}$ it holds that

(1) $a \le a$,
(2) if $a \le b$ and $b \le c$ then $a \le c$,
(3) if $b \le c$ then $a + b \le a + c$,
(4) if $a \le b$ and $\alpha \in [0, \infty)$ then $\alpha a \le \alpha b$.

We write $a < b$ if $a \le b$ and $a \ne b$. The positive cone of $\widetilde{V}$, i.e. elements $a \in \widetilde{V}$ such that $a \ge 0$, will be denoted by $\widetilde{V}_+$.

It follows that

$$a \le b \quad \Longleftrightarrow \quad -b = a - a - b \le b - a - b = -a. \tag{4.1}$$

Note also that we do not assume antisymmetry, i.e. $a \le b \le a$ does not necessarily imply that $a = b$, cf. however Definition 6.1.

When introducing a linear preorder in a gradient space, it is natural to assume that it is compatible with the notion of convergence.

Definition 4.2. A *preordered gradient space* is a gradient space $\mathcal{U} = (V, \widetilde{V}, W, \widetilde{W}, R)$ together with a linear preorder $\le$ on $\widetilde{V}$ such that if $V \ni u_i \le \psi \in \widetilde{V}$ for $i = 1, 2, \dots$, and $u_i \to u$ (in V), then $u \le \psi$.

Assume for the rest of this section that $\mathcal{U} = (V, \widetilde{V}, W, \widetilde{W}, R)$ is a preordered gradient space.

To formulate the obstacle problem for preordered gradient spaces, we proceed in analogy with the Dirichlet problem: We choose a subset $\mathcal{K}_0 \subseteq \mathrm{Sob}(\mathcal{U})$, an element $f \in \mathrm{Sob}(\mathcal{U})$, an obstacle $\psi \in \widetilde{V}$, and set

$$\mathcal{K}_{\psi,f} = \{u \in \mathrm{Sob}(\mathcal{U}) : u - f \in \mathcal{K}_0 \text{ and } u \geq \psi\}.$$

A solution to the obstacle problem with respect to $\mathcal{K}_{\psi,f}$ is then given by an element $u \in \mathcal{K}_{\psi,f}$ such that

$$\|g_u\|_W = \inf_{v \in \mathcal{K}_{\psi,f}} \|g_v\|_W \,,$$

where g_u and g_v denote the minimal gradients of u and v, respectively.

To prove the existence of a solution to the obstacle problem, we will reformulate the obstacle problem as a Dirichlet problem and use Theorem 3.2 to conclude that a minimizer exists. For this reason, we need the following result.

Lemma 4.3. *Let $\mathcal{K}_0 \subseteq \mathrm{Sob}(\mathcal{U})$ be a closed convex Poincaré set. For $f \in \mathrm{Sob}(\mathcal{U})$ and $\psi \in \widetilde{V}$ the set*

$$\widetilde{\mathcal{K}}_0(\psi, f) = \{v \in \mathcal{K}_0 : v + f \geq \psi\}$$

is a closed convex Poincaré set.

Proof. Since $\mathcal{K}_0$ is assumed to be closed, it follows from Definition 4.2 that $\widetilde{\mathcal{K}}_0(\psi, f)$ is also a closed set. Moreover, as $\mathcal{K}_0$ is a convex set, it follows that $\widetilde{\mathcal{K}}_0(\psi, f)$ is also convex (as the linear preorder respects sums and multiplication by positive real numbers). Finally, as $\widetilde{\mathcal{K}}_0(\psi, f) \subseteq \mathcal{K}_0$, and $\mathcal{K}_0$ is assumed to be a Poincaré set, also $\widetilde{\mathcal{K}}_0(\psi, f)$ is a Poincaré set. □

The existence of solutions to obstacle problems can now be obtained directly from the Dirichlet problem.

Theorem 4.4. *For any closed convex Poincaré set $\mathcal{K}_0 \subset \mathrm{Sob}(\mathcal{U})$, $f \in \mathrm{Sob}(\mathcal{U})$ and $\psi \in \widetilde{V}$, the obstacle problem with respect to $\mathcal{K}_{\psi,f}$ has at least one solution provided that $\mathcal{K}_{\psi,f} \neq \varnothing$.*

If u_1 and u_2 are solutions, then $g_{u_1} = g_{u_2}$. Moreover, if $\mathcal{K}_0$ is a linear subspace of $\mathrm{Sob}(\mathcal{U})$ and the map $u \mapsto g_u$ is linear, then $u_1 = u_2$.

Proof. In the notation of Lemma 4.3, the set $\mathcal{K}_{\psi,f}$ can be described as

$$\mathcal{K}_{\psi,f} = \{u \in \mathrm{Sob}(\mathcal{U}) : u - f \in \widetilde{\mathcal{K}}_0(\psi, f)\} = \widetilde{\mathcal{K}}_0(\psi, f) + f,$$

and since, by the same lemma, the set $\widetilde{\mathcal{K}}_0(\psi, f)$ is a closed convex Poincaré set, one can apply Theorem 3.2 to conclude that there exists at least one solution as long as $\mathcal{K}_{\psi,f} \neq \varnothing$, and that the minimal gradient of a solution is unique.

To prove the last part of the statement, we cannot use Proposition 3.4 directly, because the Poincaré set $\widetilde{\mathcal{K}}_0(\psi, f)$ is not a linear subspace. However, since it follows from the definition of $\widetilde{\mathcal{K}}_0(\psi, f)$ that (3.1) holds whenever $u_1, u_2 \in \mathcal{K}_{\psi,f}$, the proof of Proposition 3.4 applies also in this case. □

Remark 4.5. It is easily verified that Lemma 4.3 and Theorem 4.4 hold also for the following multi-obstacle problem: Given $f \in \mathrm{Sob}(\mathcal{U})$ and (possibly uncountable) sets $\Psi, \Phi \subset \widetilde{V}$, let

$$\mathcal{K}_{\Psi,\Phi,f} = \{u \in \mathrm{Sob}(\mathcal{U}) : u - f \in \mathcal{K}_0 \text{ and } \psi \leq u \leq \varphi \text{ for all } \psi \in \Psi \text{ and } \varphi \in \Phi\},$$

and find $u \in \mathcal{K}_{\Psi,\Phi,f}$ which minimizes $\|g_u\|_W$ among all $u \in \mathcal{K}_{\Psi,\Phi,f}$. Note that it is not always possible to replace Ψ by $\max_{\psi\in\Psi}\psi$ (or Φ by $\min_{\varphi\in\Phi}\varphi$), as the latter bounds need not exist or need not be optimal with respect to the ordering $\le$. See Section 6 for a further discussion on this topic.

For the multi-obstacle problem we thus obtain the following result.

Theorem 4.6. *For any closed convex Poincaré set $\mathcal{K}_0$, sets $\Psi,\Phi \subset \widetilde{V}$ and $f \in \operatorname{Sob}(\mathcal{U})$, the minimization problem*

$$\|g_u\|_W = \inf_{v\in\mathcal{K}_{\Psi,\Phi,f}} \|g_v\|_W$$

has at least one solution provided that $\mathcal{K}_{\Psi,\Phi,f} \neq \varnothing$.

If u_1 and u_2 are solutions, then $g_{u_1} = g_{u_2}$. Moreover, if $\mathcal{K}_0$ is a linear subspace of $\operatorname{Sob}(\mathcal{U})$ and the map $u \mapsto g_u$ is linear, then $u_1 = u_2$.

The assumption that $\mathcal{K}_{\psi,f}$ is nonempty in Theorem 4.4 can be rephrased in the following way.

Proposition 4.7. *Let $\mathcal{K}_0 \subseteq \operatorname{Sob}(\mathcal{U})$, $f \in \operatorname{Sob}(\mathcal{U})$ and $\psi \in \widetilde{V}$. Then $\mathcal{K}_{\psi,f} \neq \varnothing$ if and only if there exists $v \in \mathcal{K}_0$ such that $v \ge \psi - f$.*

Proof. First, assume that there is $u \in \mathcal{K}_{\psi,f}$. By setting $v = u - f$ it follows that $v \in \mathcal{K}_0$ and $v = u - f \ge \psi - f$ since $u \ge \psi$.

Conversely, assume that there exists $v \in \mathcal{K}_0$ such that $v \ge \psi - f$. Defining $u = v + f$ it follows that $u \in \operatorname{Sob}(\mathcal{U})$ (since $\mathcal{K}_0 \subseteq \operatorname{Sob}(\mathcal{U})$ and $f \in \operatorname{Sob}(\mathcal{U})$, using (G1)), $u - f = v \in \mathcal{K}_0$ and $u = v + f \ge \psi - f + f = \psi$. Hence, $u \in \mathcal{K}_{\psi,f}$. $\square$

Remark 4.8. The space $\widetilde{V}$ in a gradient space $\mathcal{U} = \bigl(V, \widetilde{V}, W, \widetilde{W}, R\bigr)$ played a prominent role of a preordered space in this section, and will also play a vital role in Section 6. It however, does not play any role in Sections 3 and 5. On the other hand, the space $\widetilde{W}$ does not play any direct role in this paper, but we have chosen to still include it for symmetry reasons and possible later applications.

5. Minimizing the Rayleigh quotient

Assume in this section that $\mathcal{U} = (V, \widetilde{V}, W, \widetilde{W}, R)$ is a gradient space.

In this section we consider another variational problem that can be treated using gradient spaces. Weak eigenfunctions of the Laplace operator can be found by minimizing the *Rayleigh quotient*

$$\operatorname{Ray}(u) = \frac{\|\nabla u\|_{L^2}}{\|u\|_{L^2}},$$

whose infimum gives the first (apart from 0) eigenvalue of the Laplace operator. If there exists a Poincaré inequality on the set over which the infimum is computed, i.e. if

$$\|u\|_{L^2} \le C\,\|\nabla u\|_{L^2}$$

for some $C > 0$, then it is clear that a positive infimum exists. To guarantee the existence of a minimizer, one can use the fact that the Sobolev space $W^{1,2}$ is compactly embedded into L^2 to find a sequence converging to a minimizer. Therefore, in the setting of gradient spaces, we shall minimize over sets which satisfy a version of the Rellich–Kondrachov theorem.

Definition 5.1. Let $\mathcal{K} \subseteq \mathrm{Sob}(\mathcal{U})$ be a cone, i.e. if $u \in \mathcal{K}$ then $\alpha u \in \mathcal{K}$ for all $\alpha > 0$. The cone $\mathcal{K}$ is a *Rellich–Kondrachov cone* if for every bounded sequence $\{u_i\}_{i=1}^{\infty} \subseteq \mathcal{K}$, such that $\{g_{u_i}\}_{i=1}^{\infty}$ is also a bounded sequence, there exists a convergent subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ which converges (in V) to an element $u \in \mathcal{K}$. Moreover, a Rellich–Kondrachov cone is *regular* if, for $u \in \mathcal{K}$, $g_u = 0$ implies that $u = 0$.

It follows directly from the definition that Rellich–Kondrachov cones are closed subsets of $\mathrm{Sob}(\mathcal{U})$.

Note that the regularity assumption is necessary for any set to be a Poincaré set. Furthermore, it is sufficient to guarantee that a Rellich–Kondrachov cone supports a Poincaré inequality.

Proposition 5.2. *A regular Rellich–Kondrachov cone is a Poincaré set.*

Proof. Let us assume that the regular Rellich–Kondrachov cone $\mathcal{K}$ is not a Poincaré set, i.e. there is no $C > 0$ such that

$$\|u\|_V \le C \|g_u\|_W$$

for all $u \in \mathcal{K}$. We will now show that, under this assumption, one may construct an element $u \in \mathcal{K}$ with $u \neq 0$ and $g_u = 0$, which contradicts the fact that $\mathcal{K}$ is regular. If there is no Poincaré inequality, then one may find a sequence $\{\tilde{u}_k\}_{k=1}^{\infty}$ in $\mathcal{K}$ such that

$$\|\tilde{u}_k\|_V \ge k \|g_{\tilde{u}_k}\|_W .$$

Since (for $\alpha > 0$) it holds that

$$\|\alpha \tilde{u}_k\|_V \ge k \|\alpha g_{\tilde{u}_k}\|_W = k \|g_{\alpha \tilde{u}_k}\|_W$$

one may construct a rescaled sequence $u_k = \tilde{u}_k / \|\tilde{u}_k\|_V$ with the same property. Thus, one obtains a sequence $\{u_k\}_{k=1}^{\infty}$ in $\mathcal{K}$ (as $\mathcal{K}$ is a cone) with $\|u_k\|_V = 1$ and $\|u_k\|_V \ge k \|g_{u_k}\|_W$, which implies that

$$\|g_{u_k}\|_W \le \frac{1}{k}.$$

Thus, $g_{u_k} \to 0$ which, in particular, implies that $\{g_{u_k}\}_{k=1}^{\infty}$ is bounded. Now, since both sequences $\{u_k\}_{k=1}^{\infty}$ and $\{g_{u_k}\}_{k=1}^{\infty}$ are bounded, and $\mathcal{K}$ is a Rellich–Kondrachov cone, we conclude that there exists a convergent subsequence $\{u_{i_k}\}_{k=1}^{\infty}$, converging to some $u \in \mathcal{K}$. Clearly, $\|u\|_V = 1$ which implies that $u \neq 0$. Moreover, $g_{u_{i_k}} \to 0$, and from (GS4) it follows that $(u, 0) \in R$, which contradicts the fact that $\mathcal{K}$ is assumed to be regular. Hence, $\mathcal{K}$ is a Poincaré set. □

Theorem 5.3. *Let $\mathcal{K} \subseteq \mathrm{Sob}(\mathcal{U})$ be a nonempty regular Rellich–Kondrachov cone. Then there exists an element $u \in \mathcal{K}$ such that*

$$\frac{\|g_u\|_W}{\|u\|_V} = \inf_{\substack{v \in \mathcal{K} \\ v \neq 0}} \frac{\|g_v\|_W}{\|v\|_V} > 0.$$

Proof. For any $u \in \mathrm{Sob}(\mathcal{U})$, such that $u \neq 0$, let

$$\mathrm{Ray}(u) = \frac{\|g_u\|_W}{\|u\|_V}.$$

For $\alpha > 0$ it follows that

$$\mathrm{Ray}(\alpha u) = \frac{\|g_{\alpha u}\|_W}{\|\alpha u\|_V} = \frac{\|\alpha g_u\|_W}{\|\alpha u\|_V} = \frac{\|g_u\|_W}{\|u\|_V} = \mathrm{Ray}(u). \tag{5.1}$$

Since $\mathcal{K}$ is a Poincaré set (by Proposition 5.2), there exists $C > 0$ such that $\operatorname{Ray}(v) \geq C$ for all $v \in \mathcal{K}$, which implies that the infimum

$$I = \inf_{\substack{v \in \mathcal{K} \\ v \neq 0}} \operatorname{Ray}(v)$$

is positive. Let $\{\tilde{u}_i\}_{i=1}^\infty$ be a minimizing sequence, i.e. a sequence such that

$$I = \lim_{i \to \infty} \operatorname{Ray}(\tilde{u}_i).$$

The normalized sequence given by $u_i = \tilde{u}_i / \|\tilde{u}_i\|_V$ clearly fulfills $\operatorname{Ray}(u_i) \geq I$ (by the definition of I as the infimum), and from (5.1) one obtains

$$I \leq \operatorname{Ray}(u_i) = \operatorname{Ray}(\tilde{u}_i),$$

which implies that $\lim_{i\to\infty} \operatorname{Ray}(u_i) = I$, i.e. $\{u_i\}_{i=1}^\infty$ is also a minimizing sequence. Moreover, it follows that $\{g_{u_i}\}_{i=1}^\infty$ is a bounded sequence since

$$I = \lim_{i \to \infty} \operatorname{Ray}(u_i) = \lim_{i \to \infty} \|g_{u_i}\|_W .$$

As $\mathcal{K}$ is a Rellich–Kondrachov cone one can find a convergent subsequence $\{u_{i_k}\}_{k=1}^\infty$, converging to some $u \in \mathcal{K}$, which must have $\|u\|_V = 1$. From the two bounded sequences $\{u_{i_k}\}_{k=1}^\infty$ and $\{g_{u_{i_k}}\}_{k=1}^\infty$ one can use Lemma 2.8 to construct convex combinations $\{\hat{u}_i\}_{i=1}^\infty$ and $\{\hat{g}_i\}_{i=1}^\infty$ which converge to u and g, respectively, with $(u, g) \in R$. Since $\hat{g}_k$, given by Lemma 2.8, are convex combinations of $g_{u_{i_k}}, g_{u_{i_{k+1}}}, \ldots$, we get

$$\|g\|_W = \lim_{i \to \infty} \|\hat{g}_i\|_W \leq \limsup_{k \to \infty} \|g_{u_{i_k}}\|_W = I.$$

It follows that

$$I \leq \operatorname{Ray}(u) = \frac{\|g_u\|_W}{\|u\|_V} = \|g_u\|_W \leq \|g\|_W \leq I$$

which shows that $\operatorname{Ray}(u) = I$. $\square$

6. Least upper bounds and lattice structures

For two real-valued functions, one may define their least upper bound by a pointwise choice of the maximum of the two function values. However, a pointwise construction is not available in the setting of (preordered) gradient spaces, and we shall investigate to what extent one may define an upper bound with the help of a minimization problem. The idea behind this approach is that the pointwise maximum of two nonnegative functions f and g minimizes the L^p-norm among all functions h such that $h \geq f$ and $h \geq g$.

More generally, for two positive elements $\psi_1, \psi_2 \in \widetilde{V}$, the aim will be to find an element $u \in V$ such that $u \geq \psi_1$, $u \geq \psi_2$ and

$$\|u\|_V = \inf_v \|v\|_V ,$$

where the infimum is taken over all $v \in V$ such that $v \geq \psi_1$ and $v \geq \psi_2$. Such an upper bound will not necessarily be a least upper bound, with respect to the linear ordering, but is only an upper bound with minimal norm. In fact, requiring that the above construction yields a least upper bound in general leads to severe restrictions on the underlying spaces in the case of noncommutative algebras.

To reach the desired results, we have to refine our notion of ordering in a gradient space. Namely, we shall assume that the norm is compatible with the ordering in the following sense.

Definition 6.1. An *ordered gradient space* $\mathcal{U} = (V, \widetilde{V}, W, \widetilde{W}, R)$ is a preordered gradient space such that V is a strictly convex Banach space and for $u, v \in V$ we have

$$0 \le u \le v \implies \|u\|_V \le \|v\|_V . \tag{6.1}$$

Assume for the rest of this section that $\mathcal{U} = (V, \widetilde{V}, W, \widetilde{W}, R)$ *is an ordered gradient space.*

It follows that the preorder $\le$ will be a partial order on V, i.e. if $u, v \in V$, $u \le v$ and $v \le u$, then $u = v$. Indeed, if $u, v \in V$, $u \le v$ and $v \le u$, then using (3) in Definition 4.1 shows that

$$0 = u - u \le v - u \le u - u = 0,$$

and hence by (6.1), $\|v - u\|_V \le \|0\|_V = 0$, so $u = v$.

Example 6.2. Note, however, that it does not follow that $\le$ is a partial order on $\widetilde{V}$. To see this, let $V = C(\mathbf{R})$ with sup-norm,

$$\widetilde{V} = \{f : f = h \text{ a.e. for some } h \in V\}$$

and let $u \le v$ if $u \le v$ pointwise a.e. as functions (where a.e. refers to the Lebesgue measure). Then $\le$ is a partial order on V but only a preorder on $\widetilde{V}$. One can clearly choose W, $\widetilde{W}$ and R (in many ways) so as to make it into an ordered gradient space.

In fact, for the results in this section, the spaces W and $\widetilde{W}$, as well as the gradient relation R, will not play any role, it is only V and $\widetilde{V}$ and the conditions imposed on them that will be involved here. However, the discussion is still important to better understand the concept of ordered gradient spaces.

The following observation will be of use to us.

Lemma 6.3. *Let* $u, v \in V$. *Then*

$$0 \le u < v \implies \|u\|_V < \|v\|_V .$$

Proof. Assume that $0 \le u < v$. By (6.1) we know that $\|u\|_V \le \|v\|_V$. Assume that $\|u\|_V = \|v\|_V$ and let $w = \frac{1}{2}(u + v)$. Then $u < w < v$ and thus, by (6.1) again, $\|u\|_V \le \|w\|_V \le \|v\|_V$. Hence $\|u\|_V = \|v\|_V = \|w\|_V = \left\|\frac{1}{2}(u + v)\right\|_V$, and since V is strictly convex $u = v$, which contradicts that $u < v$. Therefore $\|u\|_V < \|v\|_V$. $\square$

We are now ready to solve the minimization problem which constructs an upper bound of two elements in $\widetilde{V}$ with least possible norm.

Proposition 6.4. *For* $\psi_1, \psi_2 \in \widetilde{V}$ *set*

$$\Omega = \Omega(\psi_1, \psi_2) = \{u \in V : u \ge \psi_1 \text{ and } u \ge \psi_2\}.$$

If $\Omega \ne \varnothing$ *then there exists a unique element* $u \in V$ *such that*

$$\|u\|_V = \inf_{v \in \Omega} \|v\|_V .$$

Proof. Let $\{u_i\}_{i=1}^\infty$ be a minimizing sequence in Ω with

$$\lim_{i \to \infty} \|u_i\|_V = \inf_{v \in \Omega} \|v\|_V =: I.$$

Since $\{u_i\}_{i=1}^\infty$ is bounded there exists, by Corollary 2.7, a sequence $\tilde{u}_i = \sum_{j=i}^{N_i} \alpha_{ij} u_j$ converging to some u. It follows directly that $\tilde{u}_i \ge \psi_1$ and $\tilde{u}_i \ge \psi_2$ for $i = 1, 2, \ldots$.

From the definition of a preordered gradient space, it follows that $u \geq \psi_1$ and $u \geq \psi_2$, which implies that $u \in \Omega$. Furthermore, since $\tilde{u}_i$ is a convex combination of $\{u_j\}_{j=i}^{\infty}$, it is clear that

$$I \leq \|u\|_V = \lim_{i\to\infty} \|\tilde{u}_i\|_V \leq \limsup_{i\to\infty} \|u_i\|_V = I,$$

which proves that there exists a minimizer in Ω.

Next, let us prove uniqueness. Assume that u_1 and u_2 are two minimizers. In particular, $\|u_1\|_V = \|u_2\|_V = I$. Since $\frac{1}{2}(u_1 + u_2) \geq \psi_1$ and $\frac{1}{2}(u_1 + u_2) \geq \psi_2$, we get that $\frac{1}{2}(u_1 + u_2) \in \Omega$ and thus $I \leq \left\|\frac{1}{2}(u_1 + u_2)\right\|_V$. From the triangle inequality one obtains

$$I \leq \left\|\tfrac{1}{2}(u_1 + u_2)\right\|_V \leq \tfrac{1}{2}\|u_1\|_V + \tfrac{1}{2}\|u_2\|_V = I,$$

which implies that $I = \|u_1\|_V = \|u_2\|_V = \left\|\frac{1}{2}(u_1 + u_2)\right\|_V$. As V is assumed to be strictly convex, it follows that $u_1 = u_2$. $\square$

Remark 6.5. Proposition 6.4 and its proof can also be regarded as a special case of the multi-obstacle problem. For this, one defines a new gradient relation $R = \{(u, u) : u \in V\}$ with $\widetilde{W} = \widetilde{V}$ and $W = V$. The existence and uniqueness then follow directly from Theorem 4.6. Note that the minimizing sequence is bounded in this case and thus Remark 3.3 applies here. In any case, V is automatically a Poincaré set with the above choice of R.

Thus, a necessary and sufficient condition for such a minimizer to exist is that there exists at least one upper bound of ψ_1 and ψ_2 in V. Therefore, we introduce the following subset of $\widetilde{V}$.

Definition 6.6. The V*-bounded positive cone* is given as

$$B_+ = \{\psi \in \widetilde{V} : 0 \leq \psi \leq u \text{ for some } u \in V\}.$$

We also define $V_+ = V \cap B_+ = \{v \in V : v \geq 0\}$.

Note that V_+ is closed, which follows immediately from the assumption that $\mathcal{U}$ is a preordered gradient space (cf. Definition 4.2).

If $\psi_1, \psi_2 \in B_+$ then there exist $u_1, u_2 \in V$ such that $\psi_i \leq u_i$ for $i = 1, 2$. In particular, it follows that $u_1 + u_2 \geq \psi_1 + \psi_2 \geq \psi_i$ for $i = 1, 2$. Hence, there exists a unique solution to the minimization problem in Proposition 6.4 for any $\psi_1, \psi_2 \in B_+$.

Consequently, we can introduce the maximum of two V-bounded elements.

Definition 6.7. Let $\psi_1, \psi_2 \in B_+$. The unique minimizer in Proposition 6.4 is called the *maximum of* ψ_1 *and* ψ_2 and is denoted $\max(\psi_1, \psi_2)$.

Note that $\max(\psi_1, \psi_2) \in V_+$ since $\max(\psi_1, \psi_2) \in V$ and $\max(\psi_1, \psi_2) \geq \psi_1 \geq 0$. (We do not define $\max(\psi_1, \psi_2)$ unless $\psi_1, \psi_2 \in B_+$.) However, $\max(\psi_1, \psi_2)$ is not necessarily a least upper bound with respect to the ordering on V (see the example in Section 8.5). Nevertheless, it enjoys the following properties.

Lemma 6.8. *Let* $\psi_1, \psi_2 \in B_+$ *and* $u \in V$. *Then the following are true*:

(1) *if* $u \geq \psi_1$, *then* $\max(u, \psi_1) = u$,
(2) *if* $u \geq \psi_1$, $u \geq \psi_2$ *and* $u \neq \max(\psi_1, \psi_2)$, *then* $\|u\|_V > \|\max(\psi_1, \psi_2)\|_V$.

Proof. (1) Assume that $u \geq \psi_1$. Then $u \in \Omega(u, \psi_1)$. By (6.1), $\|v\|_V \geq \|u\|_V$ for $v \in \Omega(u, \psi_1)$, and hence u has minimal norm in $\Omega(u, \psi_1)$. Thus, $\max(u, \psi_1) = u$.

(2) Assume that $u \geq \psi_1$ and $u \geq \psi_2$. As $u \in \Omega(\psi_1, \psi_2)$, and $\|\max(\psi_1, \psi_2)\|_V$ is the infimum of $\|v\|_V$ among $v \in \Omega(\psi_1, \psi_2)$, we must have $\|u\|_V \geq \|\max(\psi_1, \psi_2)\|_V$. Moreover, since the minimizer is unique, we must have $\|u\|_V > \|\max(\psi_1, \psi_2)\|_V$ whenever $u \neq \max(\psi_1, \psi_2)$. □

Remark 6.9. Both for the proof of Proposition 6.4 and the one outlined in Remark 6.5, it is enough to require that $\mathcal{U}$ is a preordered gradient space with the additional requirement that V be strictly convex. However, for $u \in V_+$ it is natural to require that $\max(u, u) = u$, and this is true if and only if $\mathcal{U}$ is an ordered gradient space, which can be seen as follows: If there are u and v such that $0 \leq u \leq v$ but $\|u\|_V > \|v\|_V$, then u cannot be the solution of the minimization problem for $\max(u, u)$, and thus $\max(u, u) \neq u$. The converse direction follows from Lemma 6.8 (1).

The least upper bound property of $\max(\psi_1, \psi_2)$ is problematic, since an upper bound v of ψ_1 and ψ_2 need not be comparable to $\max(\psi_1, \psi_2)$ (see the example in Section 8.5). Here we need to be a bit more precise. If $\psi_1, \psi_2 \in B_+$, then they may have many upper bounds in $B_+ \setminus V_+$. For our purposes we will however (mainly) be interested in upper bounds that belong to V_+, and their minimality within this class. In order to clarify this, we make the following definition.

Definition 6.10. An upper bound $v \in V_+$ of $\psi_1, \psi_2 \in B_+$ is a V_+*-least upper bound* of ψ_1 and ψ_2 if $v \leq u$ for any upper bound $u \in V_+$ of ψ_1 and ψ_2.

V_+*-greatest lower bounds* are defined analogously.

Since $\leq$ is a partial order on V it follows that if a V_+-least upper bound of ψ_1 and ψ_2 exists then it is unique. Similar uniqueness holds for V_+-greatest lower bounds.

Proposition 6.11. *If $v \in V$ is such that $v \geq \psi_1 \in B_+$, $v \geq \psi_2 \in B_+$ and v is comparable to $\max(\psi_1, \psi_2)$, then $v \geq \max(\psi_1, \psi_2)$. In particular, if ψ_1 and ψ_2 have a V_+-least upper bound v, then $v = \max(\psi_1, \psi_2)$.*

Proof. If v is comparable to $\max(\psi_1, \psi_2)$ then, by definition, either $v \geq \max(\psi_1, \psi_2)$ or $v < \max(\psi_1, \psi_2)$. Since $v \in \Omega(\psi_1, \psi_2)$, the statement $v < \max(\psi_1, \psi_2)$, which implies that $\|v\|_V < \|\max(\psi_1, \psi_2)\|_V$ (by Lemma 6.3), contradicts the fact that $\max(\psi_1, \psi_2)$ minimizes the norm in $\Omega(\psi_1, \psi_2)$. Hence, $v \geq \max(\psi_1, \psi_2)$.

If, in addition, v is a V_+-least upper bound of ψ_1 and ψ_2, then one cannot have $v > \max(\psi_1, \psi_2)$ (since that would contradict the assumption that v is a *least* upper bound), and it follows that $v = \max(\psi_1, \psi_2)$. □

Let us now introduce another variational problem in order to define the minimum of two elements $\psi_1, \psi_2 \in B_+$. The idea is to minimize the (norm)-distance to elements that are lower bounds of ψ_1 and ψ_2. However, as ψ_1 and ψ_2 are not necessarily elements of V (and, in particular, $\|\psi_1 - u\|_V$ need not be defined for $u \in V$) we shall instead minimize the distance to $\max(\psi_1, \psi_2) \in V$.

Proposition 6.12. *For $\psi_1, \psi_2 \in B_+$ we set*

$$\Omega' = \Omega'(\psi_1, \psi_2) = \{v \in V : 0 \leq v \leq \psi_1 \text{ and } v \leq \psi_2\} \subset V_+.$$

Then there exists a unique element $u \in \Omega'$ such that

$$\|\max(\psi_1, \psi_2) - u\|_V = \inf_{v \in \Omega'} \|\max(\psi_1, \psi_2) - v\|_V .$$

Proof. The set Ω' contains the element 0 and is therefore always nonempty. Set $M = \max(\psi_1, \psi_2)$ and let $\{u_i\}_{i=1}^\infty$ be a minimizing sequence in Ω', i.e.

$$\lim_{i\to\infty} \|M - u_i\|_V = \inf_{v\in\Omega'} \|M - v\|_V .$$

As $\{u_i\}_{i=1}^\infty$ is a bounded sequence (due to $0 \le u_i \le \psi_1 \in B_+$), there exists (by Corollary 2.7) a sequence $\{\tilde{u}_i\}_{i=1}^\infty$ converging to some u, where each $\tilde{u}_i$ is a convex combination of $\{u_j\}_{j=i}^\infty$. It follows from the definition of a preordered gradient space that $u \in \Omega'$. Moreover,

$$I \le \|M - u\|_V = \lim_{i\to\infty} \|M - \tilde{u}_i\|_V \le \lim_{i\to\infty} \|M - u_i\|_V = I,$$

which shows that u is indeed a minimizer.

Next, let u_1 and u_2 be two minimizers fulfilling $\|M - u_1\|_V = \|M - u_2\|_V = I$. Since $u_1, u_2 \in \Omega'$ it is clear that $\frac{1}{2}(u_1 + u_2) \in \Omega'$, and one obtains

$$I \le \left\|M - \tfrac{1}{2}(u_1 + u_2)\right\|_V \le \tfrac{1}{2}\|M - u_1\|_V + \tfrac{1}{2}\|M - u_2\|_V = I,$$

which shows that

$$\|M - u_1\|_V = \|M - u_2\|_V = \tfrac{1}{2}\|M - u_1 + M - u_2\|_V .$$

As V is assumed to be strictly convex it follows that $u_1 = u_2$. $\square$

Let us now make the following definition.

Definition 6.13. For $\psi_1, \psi_2 \in B_+$ we define $\min(\psi_1, \psi_2)$ to be the (unique) minimizer in Proposition 6.12.

Our next result shows that the existence of V_+-least upper bounds implies the existence of V_+-greatest lower bounds and thus that V_+ is a lattice. Recall that a partially ordered set $(A, \le)$ is a *lattice* if every pair of elements has a greatest lower bound and a least upper bound in A.

Proposition 6.14. *If there exists a V_+-least upper bound* (*which then must be unique*) *for all pairs $u_1, u_2 \in V_+$, then V_+ is a lattice, and moreover* $\max(u_1, u_2)$ *is the V_+-least upper bound of u_1 and u_2, and* $\min(u_1, u_2)$ *is the V_+-greatest lower bound of u_1 and u_2.*

Proof. First note that $\max(u_1, u_2)$ is the V_+-least upper bound of u_1 and u_2, by Proposition 6.11. Let $v \in V_+$ be a lower bound of u_1 and u_2 and set

$$\hat{v} = \max(v, \min(u_1, u_2)).$$

Since u_1 and u_2 are upper bounds for both v and $\min(u_1, u_2)$, we have $\hat{v} \le u_i$ for $i = 1, 2$ as $\hat{v}$ is the least such upper bound (by Proposition 6.11 again). Hence $\max(u_1, u_2) - \hat{v} \ge 0$. Using (4.1) and $\hat{v} \ge \min(u_1, u_2)$, we then conclude that

$$0 \le \max(u_1, u_2) - \hat{v} \le \max(u_1, u_2) - \min(u_1, u_2),$$

from which it follows that

$$\|\max(u_1, u_2) - \hat{v}\|_V \le \|\max(u_1, u_2) - \min(u_1, u_2)\|_V ,$$

as $\mathcal{U}$ is an ordered gradient space. Since $\min(u_1, u_2)$ is the unique minimizer of the above norm (by Proposition 6.12), we must have $\hat{v} = \min(u_1, u_2)$. Hence,

$$\min(u_1, u_2) = \max(v, \min(u_1, u_2)),$$

which implies that $\min(u_1, u_2) \ge v$. Thus u_1 and u_2 have a V_+-greatest lower bound which equals $\min(u_1, u_2)$, and hence V_+ is a lattice. $\square$

We can now obtain the following characterization.

Theorem 6.15. *The following are equivalent*:

(a) *there is a V_+-least upper bound for all $u_1, u_2 \in V_+$,*
(b) $\max(u_1, u_2)$ *is the V_+-least upper bound for all $u_1, u_2 \in V_+$,*
(c) *V_+ is a lattice,*
(d) *there is a V_+-greatest lower bound for all $u_1, u_2 \in V_+$,*
(e) $\min(u_1, u_2)$ *is the V_+-greatest lower bound for all $u_1, u_2 \in V_+$.*

Proof. (a) $\Rightarrow$ (b) This follows from Proposition 6.11.

(b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (e) This is the conclusion of Proposition 6.14.

(c) $\Rightarrow$ (a) This follows from the definition of lattice.

(e) $\Rightarrow$ (d) This is trivial.

To complete the proof we show that (d) $\Rightarrow$ (b). Let $u_1, u_2 \in V_+$ and $u = \max(u_1, u_2) \in V_+$. Also let $\tilde{u} \in V_+$ be an arbitrary upper bound of u_1 and u_2. Let v be the V_+-greatest lower bound of u and $\tilde{u}$, which exists by assumption. As u_i is also a common lower bound, we have $u_i \le v$, $i = 1, 2$. Hence v is an upper bound of u_1 and u_2. As u is the upper bound with least norm, we have $\|v\|_V \ge \|u\|_V$. Since also $v \le u$, Lemma 6.3 implies that $v = u$. Thus $u = v \le \tilde{u}$, showing that $u = \max(u_1, u_2)$ is the V_+-least upper bound of u_1 and u_2. As u_1 and u_2 were arbitrary, we have shown (b). $\square$

A natural question to ask is whether V (and not only V_+) is a lattice in this setting? As the ordering is linear, one can show that as soon as every pair of elements in V has a common upper bound (in V), it follows that V is a lattice; clearly this is also a necessary condition. One can refine this a bit, and in order to do so let us introduce the linear subspace (see Lemma 6.19)

$$V_0 := \{u \in V : u \le v \text{ for some } v \in V_+\} = \{u - v : u, v \in V_+\}$$

of V. We then have the following results (whose proofs we postpone to the end of this section).

Proposition 6.16. *V_0 is a lattice if and only if V_+ is a lattice. In this case, for all $u_1, u_2 \in V_+$, their V_+-least upper bound* $\max(u_1, u_2)$ *is even least among all upper bounds in V.*

Similarly, the V_+-greatest lower bound $\min(u_1, u_2)$ *is greatest also among all lower bounds of u_1 and u_2 in V.*

In particular, it follows that one can replace "V_+-least" and "V_+-greatest" by "V-least" and "V-greatest" (with the obvious interpretation) in Theorem 6.15 (while leaving the other V_+'s) to produce four more equivalent statements.

Theorem 6.17. *V is a lattice if and only if $V = V_0$ and V_+ is a lattice.*

Note that $V = V_0$ if and only if every element in V has an upper bound in V_+, or equivalently, if every pair of elements in V has a common upper bound in V.

The condition $V = V_0$ cannot be dropped, as seen by the following example.

Example 6.18. Let $V = \widetilde{V} = \mathbf{C}$ and say that $u \le v$ if $\operatorname{Re} u \le \operatorname{Re} v$ and $\operatorname{Im} u = \operatorname{Im} v$. Then $V_+ = \{\lambda \in \mathbf{R} : \lambda \ge 0\}$ and $V_0 = \mathbf{R} \ne V$. Here V_+ and V_0 are lattices, but V is not.

We do not know if V_0 is always closed, but we next show that V_0 is indeed a linear subspace.

Lemma 6.19. *V_0 is a linear subspace.*

Proof. Let $u, v \in V_0$ and $\alpha \geq 0$. Then it is rather obvious that $u+v, \alpha u \in V_0$. The only nonobvious fact we need to show is that $-u \in V_0$. As $u \in V_0$, there is $w \in V_+$ such that $u \leq w$. Then $0 \leq w-u \in V$ and $w \geq 0$. Hence $-u \leq w-u \in V_+$. □

Proof of Proposition 6.16. First assume that V_+ is a lattice, and let $u_1, u_2 \in V_0$. We need to show that u_1 and u_2 have a V_0-least upper bound. By Lemma 6.19, $-u_i \in V_0$ and thus it has an upper bound $v_i \in V_+ \subset V_0$, $i = 1, 2$. It follows that $w := -v_1 - v_2 \in V_0$ is a common lower bound of u_1 and u_2. Hence, $0 \leq u_i - w \in V_+$, $i = 1, 2$. Then $z = \max(u_1 - w, u_2 - w)$ is the V_+-least upper bound of $u_1 - w$ and $u_2 - w$, by Theorem 6.15, as V_+ is assumed to be a lattice. Any upper bound in V of $u_1 - w$ and $u_2 - w$ necessarily belongs to V_+, and thus z is also least among all upper bounds of $u_1 - w$ and $u_2 - w$ in V. It follows that $w + z$ is a V-least upper bound of u_1 and u_2. As $w + z \in V_0$, it is least among all upper bounds in V_0 as well. Applying this to $-u_1$ and $-u_2$ shows that u_1 and u_2 also have a V_0-greatest lower bound, and thus V_0 is a lattice.

Next, assume that V_0 is a lattice and let $u_1, u_2 \in V_+$. Then u_1 and u_2 have a V_0-least upper bound w and a V_0-greatest lower bound z. Any upper bound of u_1 (in V) must belong to V_+ and thus w is a V_+-least upper bound of u_1 and u_2. Moreover, 0 is a common lower bound of u_1 and u_2, so $z \geq 0$, as it is greatest, and thus z is a V_+-greatest lower bound. Hence V_+ is a lattice.

Finally, let $u_1, u_2 \in V_+$ and assume that V_+ (or equivalently V_0) is a lattice. By the above, the V_+-least upper (greatest lower) bound of u_1 and u_2 is V_0-least (greatest) as well. It follows that it is also V-least (greatest), since every upper (lower) bound $v \in V$ of u_1 and u_2 satisfies $v \geq u_1 \geq 0$ ($v \leq u_1 \in V_+$) and thus necessarily belongs to V_0. □

Proof of Theorem 6.17. First, assume that V is a lattice. Then $u \in V$ and 0 have a V-least upper bound w which must belong to V_+, so $u \in V_0$, i.e. $V_0 = V$. Moreover, V_0 is thus a lattice by assumption, and hence, by Proposition 6.16, V_+ is also a lattice.

Conversely, if V_+ is a lattice and $V = V_0$, then $V = V_0$ is also a lattice by Proposition 6.16 again. □

7. Examples of gradient spaces. I. Sobolev spaces

In this and the next section we shall present a collection of examples that serve as a justification for the abstract gradient spaces we have introduced. As we will see, several known cases of generalized gradients are included and, in some cases, our framework leads to extensions of previously known results, as well as a few new results. We also show that higher-order operators, such as the Laplacian, can be considered as "gradients" in our framework. Moreover, we demonstrate explicitly that noncommutative algebras can be included, by studying finite-dimensional matrix algebras as well as infinite-dimensional operator algebras.

7.1. **Sobolev spaces on unweighted $\mathbf{R}^n$.** Let $\mathcal{M}(\mathbf{R}^n)$ and $\mathcal{M}(\mathbf{R}^n, \mathbf{R}^n)$ be the sets of a.e.-equivalence classes (with respect to the Lebesgue measure) of measurable functions from $\mathbf{R}^n$ to $\mathbf{R}$ and from $\mathbf{R}^n$ to $\mathbf{R}^n$, respectively. Let $1 < p < \infty$. We introduce a relation R so that

$$\mathcal{U} = (L^p(\mathbf{R}^n), \mathcal{M}(\mathbf{R}^n), L^p(\mathbf{R}^n, \mathbf{R}^n), \mathcal{M}(\mathbf{R}^n, \mathbf{R}^n), R) \tag{7.1}$$

is a gradient space, where $L^p(\mathbf{R}^n, \mathbf{R}^n)$ is the set of vector-valued L^p functions from $\mathbf{R}^n$ to $\mathbf{R}^n$. Here we let

$$(u, \nabla u) \in R$$

if ∇u is the distributional gradient of u, defined by

$$\int_{\mathbf{R}^n} u(x)\nabla\varphi(x)\,dx = -\int_{\mathbf{R}^n} \nabla u(x)\varphi(x)\,dx \quad \text{for all } \varphi \in C_0^\infty(\mathbf{R}^n), \tag{7.2}$$

where $C_0^\infty(\mathbf{R}^n)$ is the space of all infinitely differentiable functions with compact support in $\mathbf{R}^n$. Then $\mathrm{Sob}(\mathcal{U})$ becomes the usual Sobolev space $W^{1,p}(\mathbf{R}^n)$ and since $L^p(\mathbf{R}^n)$ (for $1 < p < \infty$) is uniformly convex, properties (GS1) and (GS2) hold. It follows immediately from (7.2) that $\nabla(-u) = -\nabla u$, and thus (GS3) holds. It is also straightforward that (7.2) is preserved under taking limits as $u_j \to u$ in $L^p(\mathbf{R}^n)$ and $\nabla u_j \to v$ in $L^p(\mathbf{R}^n, \mathbf{R}^n)$, i.e. $(u, v) \in R$. Hence, (GS4) is fulfilled and we conclude that $\mathcal{U}$ is a gradient space.

A (linear) partial order on $\mathcal{M}(\mathbf{R}^n)$ is introduced by writing $f \geq 0$ if the set $\{x \in \mathbf{R}^n : f(x) < 0\}$ has Lebesgue measure zero. One also easily verifies that the requirements in Definitions 4.2 and 6.1 are satisfied. We may thus conclude that (7.1) is an ordered gradient space with respect to the standard (almost everywhere) ordering of functions.

It is well known that if $\Omega \subset \mathbf{R}^n$ is a bounded open set then

$$\|u\|_{L^p(\Omega)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega)} \quad \text{for all } u \in C_0^\infty(\Omega),$$

see e.g. Theorem 2.4.1 in Ziemer [25]. Thus, the closure $W_0^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ in $W^{1,p}(\mathbf{R}^n)$ is a Poincaré set. By Theorem 2.5.1 in [25], it is also a regular Rellich–Kondrachov cone.

With respect to the above gradient relation, the corresponding Dirichlet problem becomes the p-Laplace equation

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

with the boundary data $u = f \in W^{1,p}(\mathbf{R}^n)$ on $\partial\Omega$, while the Rayleigh quotient gives rise to the eigenvalue problem $\Delta_p u - \lambda u^{p-1} = 0$ for $0 \leq u \in W_0^{1,p}(\Omega)$.

Another possibility is to let

$$\widetilde{\mathcal{U}} = (L^p(\mathbf{R}^n), \mathcal{M}(\mathbf{R}^n), L^p(\mathbf{R}^n), \mathcal{M}(\mathbf{R}^n), \widetilde{R})$$

with $(u, g) \in \widetilde{R}$ if $g \geq |\nabla u|$ a.e. (note that we need an inequality for property (G1) to hold). This leads to the same Sobolev space $\mathrm{Sob}(\widetilde{\mathcal{U}}) = W^{1,p}(\mathbf{R}^n)$. It is easily seen that all our axioms are satisfied also in this case and that we obtain an ordered gradient space (with almost everywhere ordering). This definition of gradient relation is less orthodox, but is more in line with the metric space definitions given in Sections 7.3–7.6 below. Let us however first consider weighted $\mathbf{R}^n$.

7.2. **Weighted $\mathbf{R}^n$.** Let $1 < p < \infty$ and let $d\mu = w(x)\,dx$ be a p-admissible weight, see Heinonen–Kilpeläinen–Martio [7]. Here we let

$$\mathcal{U} = (L^p(\mathbf{R}^n, \mu), \mathcal{M}(\mathbf{R}^n), L^p(\mathbf{R}^n, \mathbf{R}^n, \mu), \mathcal{M}(\mathbf{R}^n, \mathbf{R}^n), R)$$

with $(u, v) \in R$ if v is the gradient of u defined in Section 1.9 in [7], i.e. there is a sequence $\varphi_i \in C^\infty(\mathbf{R}^n)$ such that both $\varphi_i \to u$ in $L^p(\mathbf{R}^n, \mu)$ and $\nabla\varphi_i \to v$ in $L^p(\mathbf{R}^n, \mathbf{R}^n, \mu)$. (The gradient depends on p and μ, but for locally Lipschitz functions it coincides with the usual distributional gradient.) We thus obtain $\mathrm{Sob}(\mathcal{U}) = H^{1,p}(\mathbf{R}^n; \mu)$, in the notation of [7]. If μ is the Lebesgue measure, then

$H^{1,p}(\mathbf{R}^n;\mu)$ is the unweighted Sobolev space $W^{1,p}(\mathbf{R}^n)$ from Section 7.1. A natural partial order on $\mathrm{Sob}(\mathcal{U})$ is as before given by μ-a.e. pointwise inequality of functions.

As in the unweighted case, for a bounded open set $\Omega \subset \mathbf{R}^n$, the closure $H_0^{1,p}(\Omega;\mu)$ of $C_0^\infty(\Omega)$ in $H^{1,p}(\mathbf{R}^n;\mu)$ will be a Poincaré set and a regular Rellich–Kondrachov cone. The corresponding Dirichlet problem will be

$$\min \int_\Omega |\nabla u|^p w(x)\,dx$$

among all $u \in H^{1,p}(\mathbf{R}^n;\mu)$ with $u - f \in H_0^{1,p}(\Omega;\mu)$. Here, ∇u stands for the gradient defined above. One easily verifies that the minimization problem is equivalent to the weighted p-Laplace equation $\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = 0$, see Chapter 5 in [7], and that the Rayleigh quotient leads to the weighted eigenvalue problem $\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) - \lambda u^{p-1} w(x) = 0$.

As in the unweighted situation, we can alternatively let

$$\widetilde{\mathcal{U}} = (L^p(\mathbf{R}^n,\mu), \mathcal{M}(\mathbf{R}^n), L^p(\mathbf{R}^n,\mu), \mathcal{M}(\mathbf{R}^n), \widetilde{R}) \tag{7.3}$$

with $(u,g) \in \widetilde{R}$ if $g \geq |\nabla u|$ a.e. Then $\mathrm{Sob}(\widetilde{\mathcal{U}}) = H^{1,p}(\mathbf{R}^n;\mu)$ as before, and the above minimization problems will be the same as in the vector-valued case. One can also consider similar gradient spaces on open subsets of $\mathbf{R}^n$. In all cases it is easily seen that all our axioms are satisfied and that we obtain ordered gradient spaces.

Another nonstandard choice of a gradient relation in unweighted $\mathbf{R}^n$ is

$$\widehat{R} = \{(u,g) : u \in W^{1,p}(\mathbf{R}^n) \text{ and } g \geq M|\nabla u| \text{ a.e.}\},$$

where $M|\nabla u|$ is the Hardy–Littlewood maximal function of ∇u. This choice is related to the Hajłasz gradient in Section 7.5 below and leads to yet another Dirichlet problem. Similarly, on weighted $\mathbf{R}^n$, the weighted maximal function

$$M_\mu v(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |v|\,d\mu$$

provides a gradient relation.

For other possible choices of gradient relations leading to different minimization problems and partial differential equations see Section 8.2 below.

7.3. **Newtonian Sobolev spaces on metric spaces.** In order to see how to fit these spaces into our theory of gradient spaces we first need to give a (very brief) introduction to the theory of Newtonian spaces.

Let $1 < p < \infty$ and let $X = (X,d,\mu)$ be a metric space equipped with a metric d and a Borel regular measure μ, which is positive and finite on all balls. A measurable function $g : X \to [0,\infty]$ is a *p-weak upper gradient* of an everywhere defined function $u : X \to [-\infty,\infty]$ if for p-almost all nonconstant rectifiable curves γ in X,

$$|u(x) - u(y)| \leq \int_\gamma g\,ds, \tag{7.4}$$

where x and y are the endpoints of γ, and the integration over γ is with respect to the arc length ds. Here p-almost all means that there exists $\rho \in L^p(X)$ such that $\int_\gamma \rho\,ds = \infty$ for all curves γ failing (7.4), see e.g. Chapter 1 in Björn–Björn [1] for this and the other basic Newtonian theory needed here.

Moreover, a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if (7.4) holds for *all* nonconstant rectifiable curves. However, if we had based our theory upon upper gradients, then (GS4) would have failed, and therefore the p-weak upper gradients will be the ones that are of interest to us.

Upper gradients were introduced by Heinonen and Koskela [8], while Koskela and MacManus [12] introduced the p-weak upper gradients (because of the problem mentioned above with upper gradients and (GS4)). Shanmugalingam [18] defined the Newtonian spaces based on these notions (they can equivalently be defined using either upper gradients or p-weak upper gradients) and showed that they are always Banach spaces.

Note that u above is required to be everywhere defined for the definition of (p-weak) upper gradients to make sense. Indeed, it is easily verified that if $\tilde{u} = u$ μ-a.e. and g is a p-weak upper gradient of u, then g need not be a p-weak upper gradient of $\tilde{u}$. (In $\mathbf{R}^2$, let e.g. $u \equiv 0$ and $\tilde{u} = \chi_{\mathbf{R}\times\{0\}}$, i.e. the characteristic function of $\mathbf{R} \times \{0\}$.) However, as we want V and W to be normed (not just seminormed) spaces, the easiest approach is to consider μ-a.e. equivalence classes.

We therefore let $\mathcal{M}(X)$ be the set of μ-a.e. equivalence classes of measurable functions from X to $[-\infty, \infty]$, and set

$$\mathcal{U} = (L^p(X), \mathcal{M}(X), L^p(X), \mathcal{M}(X), R),$$

where $([u], [g]) \in R$ if there are everywhere defined representatives $\tilde{u} \in [u]$ and $\tilde{g} \in [g]$ such that $\tilde{g}$ is a p-weak upper gradient of $\tilde{u}$. Note, however, that not all representatives of $[u]$ have p-weak upper gradients in $L^p(X)$.

The space $\operatorname{Sob}(\mathcal{U})$ becomes the Newtonian Sobolev space. Strictly speaking $\operatorname{Sob}(\mathcal{U}) = \widehat{N}^{1,p}(X)/\sim$ in the notation of [1] (where $\sim$ is the μ-a.e.-equivalence relation), while the Newtonian space $N^{1,p}(X)$ considered therein consists of all pointwise defined $u \in L^p(X)$ which have p-weak upper gradients in $L^p(X)$. If $X = \mathbf{R}^n$ (with Euclidean distance) and $d\mu = w\,dx$ is a p-admissible weight, then $\operatorname{Sob}(\mathcal{U})$ coincides with $H^{1,p}(\mathbf{R}^n; \mu)$ from Section 7.2, see Appendix A.2 in [1]. If μ is the Lebesgue measure, then $\operatorname{Sob}(\mathcal{U}) = W^{1,p}(\mathbf{R}^n)$, see Shanmugalingam [18].

That axioms (G1), (G2) and (GS1)–(GS3) hold follows from results in Chapter 1 of [1], while (GS4) follows from Proposition 2.3 in [1]. It is also clear (due to the monotonicity of integration) that $\mathcal{U}$ is an ordered gradient space and a lattice (with the μ-a.e. ordering). Moreover, the maximum $\max(u, v)$ is then the μ-a.e. pointwise maximum.

The minimal gradient that we obtained in Theorem 2.11 is in this case usually called the minimal p-weak upper gradient and it is not only norm-minimal but also pointwise minimal μ-a.e. and local in the sense that it is zero μ-a.e. on every level set $\{x \in X : u(x) = c\}$, see Chapter 2 in [1].

Standard assumptions in the theory of Newtonian spaces on metric spaces are that the underlying measure μ is *doubling* and supports a *p-Poincaré inequality*, i.e. there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B = B(x, r)$ we have $\mu(2B) \leq C\mu(B)$, and for all integrable functions u on X and all (p-weak) upper gradients g of u,

$$\fint_B |u - u_B|\,d\mu \leq Cr\left(\fint_{\lambda B} g^p\,d\mu\right)^{1/p}, \tag{7.5}$$

where $f_B := \fint_B f\,d\mu := \int_B f\,d\mu/\mu(B)$ and $\lambda B = B(x, \lambda r)$. Here, for simplicity, we also assume that X is unbounded. These assumptions imply that the Friedrichs'

inequality

$$\int_E |u|^p \, d\mu \le C_E \int_E g_u^p \, d\mu \tag{7.6}$$

holds for all u in

$$\widehat{\mathcal{K}}_0(E) := \{[u] \in \mathrm{Sob}(\mathcal{U}) : u = 0 \text{ on } X \setminus E\}, \tag{7.7}$$

where E is an arbitrary bounded measurable subset of X. Thus, the space $\widehat{\mathcal{K}}_0(E)$ is a (closed convex) Poincaré set for any bounded measurable set $E \subset X$, and the theory from Sections 2–4 can be applied.

Usually one considers the slightly smaller zero-Sobolev space

$$\mathcal{K}_0(E) := \{[u] : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus E\} \subset \widehat{\mathcal{K}}_0(E). \tag{7.8}$$

If G is open, and X is weighted $\mathbf{R}^n$ (as in Section 7.2), then $\mathcal{K}_0(G) = H_0^{1,p}(G;\mu)$, while $\widehat{\mathcal{K}}_0(G)$ can be larger; e.g. if $G = B \setminus K$, where B is a ball and $K \subset B$ is a compact set with zero measure and positive capacity, then

$$\widehat{\mathcal{K}}_0(G) = H_0^{1,p}(B;\mu) \supsetneq H_0^{1,p}(G;\mu) = \mathcal{K}_0(G).$$

For nonopen E the space $\mathcal{K}_0(E)$ is essentially the space $N_0^{1,p}(E)$, cf. the study of Dirichlet and obstacle problems on nonopen sets in Björn–Björn [2]. The Dirichlet problem in this setting was first considered by Shanmugalingam [19], and extensively studied since then. See Björn–Björn [1] for further references.

Since the minimal p-weak upper gradient is not a linear operation (as $g_{-u} = g_u$), Proposition 3.4 cannot be applied in this case. Nevertheless, uniqueness of solutions of the Dirichlet problem is proved by different methods under the assumption of a p-Poincaré inequality, see Cheeger [3, Theorem 7.14] or Björn–Björn [2, Theorem 7.2].

It is also known that the p-Poincaré inequality implies the Sobolev embedding $\widehat{\mathcal{K}}_0(E) \to L^q(E)$ for some $q > p$, and that the embedding $\widehat{\mathcal{K}}_0(E) \to L^p(E)$ is compact, see Björn–Björn [1, Theorem 5.51] and Hajłasz–Koskela [6, Theorems 5.1 and 8.1]. Thus, $\mathcal{K}_0(E)$ and $\widehat{\mathcal{K}}_0(E)$ are regular Rellich–Kondrachov cones, which makes it possible to also solve the "eigenvalue problems"

$$\min_{u \in \mathcal{K}_0(E)} \frac{\|g_u\|_{L^p(X)}}{\|u\|_{L^p(X)}} \quad \text{and} \quad \min_{u \in \widehat{\mathcal{K}}_0(E)} \frac{\|g_u\|_{L^p(X)}}{\|u\|_{L^p(X)}},$$

see e.g. Latvala–Marola–Pere [13].

7.4. **Newtonian spaces based on Banach function lattices.** In the above theory of Newtonian spaces one can replace the L^p space by another space.

A vector space $Y \subset \mathcal{M}(X)$ of (μ-a.e. equivalence classes of) measurable functions on a metric space (X, d, μ) (as in Section 7.3) is a *Banach function lattice* if the following axioms hold:

- (P0) $\|\cdot\|_Y$ determines Y, i.e. $Y = \{u \in \mathcal{M}(X) : \|u\|_Y < \infty\}$;
- (P1) $\|\cdot\|_Y$ is a norm;
- (P2) the lattice property holds, i.e. if $|u| \le |v|$ μ-a.e. then $\|u\|_Y \le \|v\|_Y$;
- (RF) the Riesz–Fischer property holds, i.e. if $u_n \ge 0$ μ-a.e. for all n, then

$$\Bigl\| \sum_{n=1}^\infty u_n \Bigr\|_Y \le \sum_{n=1}^\infty \|u_n\|_Y.$$

Note that a Banach function space is a more restrictive concept requiring some further axioms, see the discussion in Malý [15].

Let us additionally assume that Y is reflexive, and that the norm is strictly convex. We then introduce the gradient space

$$\mathcal{U} = (Y, \mathcal{M}(X), Y, \mathcal{M}(X), R),$$

where $([u],[g]) \in R$ if there are representatives $\tilde{u} \in [u]$ and $\tilde{g} \in [g]$ such that $\tilde{g}$ is a Y-weak upper gradient of $\tilde{u}$, i.e. that (7.4) holds for Y-almost all curves γ. For the exact definition of Y-weak upper gradients see [15], where the basic theory is developed. That axioms (G1), (G2) and (GS1)–(GS3) hold also follows from results in [15], while (GS4) follows from Proposition 5.6 in Malý [14]. Moreover, the gradient space is ordered (with the μ-a.e. ordering).

The minimal gradient that we obtained in Theorem 2.11 is in this case usually called the minimal Y-weak upper gradient. Also here it is pointwise minimal μ-a.e. and local, see [14]. Note however that in [14] and [15], Y is not assumed to be reflexive and the norm $\|\cdot\|_Y$ need not be strictly convex.

One can define $\mathcal{K}_0(E)$ and $\widehat{\mathcal{K}}_0(E)$ similarly as in (7.7) and (7.8). They are closed, by e.g. Corollary 7.2 in [15], and clearly convex. Whenever they are Poincaré sets our results in Sections 3 and 4 show the existence of solutions to the Dirichlet and obstacle problems. (We are not aware of any earlier such results in the Newtonian theory beyond L^p spaces.)

7.5. **The Hajłasz gradient.** The theory of Hajłasz gradients, as introduced by Hajłasz in [5], is similar to that of (p-weak) upper gradients in that it also applies to arbitrary metric spaces and that the gradient is scalar (rather than vector-valued) and the operation of minimal Hajłasz gradient is not linear. There are, however, several substantial differences, such as lack of locality, see below.

Let (X, d, μ) be a metric space as in Section 7.3. A Borel function $h : X \to [0,\infty]$ is a *Hajłasz gradient* of a function $u : X \to [-\infty,\infty]$ if for all $x, y \in X \setminus Z$, where $\mu(Z) = 0$, we have that

$$|u(x) - u(y)| \le d(x,y)(h(x) + h(y)). \tag{7.9}$$

As before, to obtain an ordered gradient space we choose $1 < p < \infty$ and let

$$\mathcal{U} = (L^p(X), \mathcal{M}(X), L^p(X), \mathcal{M}(X), R).$$

This time $(u, h) \in R$ if h is a Hajłasz gradient of u (which is easily seen to be well-defined on μ-a.e. equivalence classes of functions).

The Sobolev space $\mathrm{Sob}(\mathcal{U})$ then coincides with the Hajłasz space $M^{1,p}(X)$, consisting of all $u \in L^p(X)$ with a Hajłasz gradient in $L^p(X)$. This almost always (except for some pathological situations) makes $\mathrm{Sob}(\mathcal{U})$ to be strictly smaller that $L^p(X)$. In contrast, if X contains no nonconstant rectifiable curves (e.g. $X = \mathbf{R} \setminus \mathbf{Q}$ or $X = \mathbf{R}$ with the snowflaked metric $d(x,y) = |x-y|^\alpha$, $0 < \alpha < 1$), then $\widehat{N}^{1,p}(X)/\sim$ equals $L^p(X)$. Lemma 4.7 and Theorem 4.8 in Shanmugalingam [18] show that one always has $M^{1,p}(X) \subset \widehat{N}^{1,p}(X)/\sim$ and that 4 times a Hajłasz gradient is an upper gradient. Moreover, if μ is doubling and X satisfies a q-Poincaré inequality for upper gradients for some $q < p$, then $M^{1,p}(X) = \widehat{N}^{1,p}(X)/\sim$. (Note that if X is complete, μ is doubling and X satisfies an (upper gradient) p-Poincaré inequality, then by Keith–Zhong [11] it also satisfies a q-Poincaré inequality for some $q < p$.)

It is easily verified that axioms (G1), (G2) and (GS1)–(GS3) hold for Hajłasz gradients. That (GS4) is satisfied follows from the fact that every L^p-convergent sequence has a μ-a.e. converging subsequence. Moreover, the gradient space is ordered (with the μ-a.e. ordering). Thus, the results in Section 2 apply to Hajłasz spaces and show in particular that they are Banach spaces and that there is a unique minimal Hajłasz gradient. These properties were originally proved in Theorems 2 and 3 in [5].

Next, Hajłasz functions *always* satisfy the 1-Poincaré inequality, as is easily seen by twice integrating (7.9) over a ball B with respect to x and y. As for the Friedrichs' inequality (7.6), a similar repeated integration over B and $2B \setminus B$ gives for all $u \in M^{1,p}(X)$ vanishing outside a ball $B = B(x,r)$,

$$\begin{aligned}
\fint_B |u|^p \, d\mu &= \fint_{2B\setminus B} \fint_B |u(x) - u(y)|^p \, d\mu(x) \, d\mu(y) \\
&\le \fint_{2B\setminus B} \fint_B d(x,y)^p (h(x) + h(y))^p \, d\mu(x) \, d\mu(y) \\
&\le C' r^p \biggl(\fint_B h(x)^p \, d\mu(x) + \fint_{2B\setminus B} h(y)^p \, d\mu(y) \biggr) \\
&\le C'' r^p \fint_{2B} h^p \, d\mu,
\end{aligned}$$

provided that μ is doubling and reverse doubling for B, i.e. that

$$(1+\varepsilon)\mu(B) \le \mu(2B) \le C\mu(B)$$

for some $\varepsilon, C > 0$. Thus,

$$\|u\|_{L^p(X)}^p = \int_B |u|^p \, d\mu \le C_B \int_{2B} h^p \, d\mu \le C_B \|h\|_{L^p(X)}^p.$$

Note here that the last integral cannot be taken only over B, since the Hajłasz gradient lacks locality, i.e. $u = 0$ in some set does not imply that the minimal Hajłasz gradient $h_u = 0$ in that set. On the other hand, locality holds for minimal weak upper gradients based on curves. Also contrary to the upper gradient case, the minimal Hajłasz gradients are only norm-minimal, and not μ-a.e pointwise minimal, see Example B.1 in [1].

Nevertheless, our results in Sections 3 and 4 show the existence of solutions to the Dirichlet and obstacle problems. As far as we know, these problems have not been considered for the Hajłasz spaces before.

7.6. **Gradients from Poincaré inequalities.** Another possibility to define a gradient relation is through Poincaré inequalities; namely, we say that $(u,k) \in R$ if for all balls $B = B(x,r) \subset X$,

$$\fint_B |u - u_B| \, d\mu \le r \fint_{\lambda B} k \, d\mu, \tag{7.10}$$

where u_B is the integral average as before, and $\lambda \ge 1$ is some fixed constant. We also let

$$\mathcal{U} = (L^p(X), \mathcal{M}(X), L^p(X), \mathcal{M}(X), R),$$

$1 < p < \infty$.

It is clear that (G1) and (G2) hold. (Note, however, that for (G2) it is important that we use the 1-Poincaré inequality in (7.10), since with a p-Poincaré inequality in (7.10) the subadditivity is not at all clear.) Also, (GS1)–(GS4) are clearly satisfied, since (7.10) is preserved under taking L^p-limits.

The space $\operatorname{Sob}(\mathcal{U})$ consists of all $u \in L^p(X)$ such that (7.10) holds for some *Poincaré gradient* $k \in L^p(X)$. Theorem 2.11 then implies that there exists a μ-a.e. unique minimal Poincaré gradient k_u satisfying (7.10).

To solve the Dirichlet and obstacle problems we need Poincaré sets. An application of Hölder's inequality to (7.10) implies the p-Poincaré inequality (7.5) (with g replaced by k). Assume that μ is doubling. We can now use Theorem 5.1 from Hajłasz–Koskela [6] which implies the (q,p)-Poincaré inequality

$$\left(\fint_B |u-u_B|^q \, d\mu\right)^{1/q} \le Cr\left(\fint_{5\lambda B} k_u^p \, d\mu\right)^{1/p} \tag{7.11}$$

for some $q>p$, and in particular the (p,p)-Poincaré inequality. Now let $E \subset X$ be a bounded and measurable set, and fix a ball B such that $E \subset B$ and $\mu(B \setminus E) > 0$. Let $\mathcal{K}_0$ consist of all $u \in \operatorname{Sob}(\mathcal{U})$ such that $u = 0$ μ-a.e. in $X \setminus E$. Then (7.11) with $q = p$ implies that for every $u \in \mathcal{K}_0$,

$$\begin{aligned}\left(\fint_B |u|^p \, d\mu\right)^{1/p} &\le \left(\fint_B |u-u_B|^p \, d\mu\right)^{1/p} + |u_B| \\ &\le Cr\left(\fint_{5\lambda B} k_u^p \, d\mu\right)^{1/p} + |u_B|. \end{aligned} \tag{7.12}$$

Moreover, Hölder's inequality and the fact that $u=0$ μ-a.e. outside E yield

$$|u_B| \le \fint_B |u|\chi_E \, d\mu \le \left(\fint_B |u|^p \, d\mu\right)^{1/p}\left(\frac{\mu(E)}{\mu(B)}\right)^{1-1/p} \le \theta\left(\fint_B |u|^p \, d\mu\right)^{1/p},$$

where $\theta \in (0,1)$. Inserting this into (7.12) and subtracting from both sides yields

$$(1-\theta)\left(\fint_B |u|^p \, d\mu\right)^{1/p} \le Cr\left(\fint_{5\lambda B} k_u^p \, d\mu\right)^{1/p}.$$

From this and the doubling property of μ we conclude that

$$\|u\|_V^p = \int_B |u|^p \, d\mu \le C_E \int_{5\lambda B} k_u^p \, d\mu \le C_E \|k_u\|_W^p,$$

i.e. that $\mathcal{K}_0$ is a Poincaré set. Proposition 3.4 now makes it possible to solve the Dirichlet problem $\min \|k_u\|_W$ among all $u \in \operatorname{Sob}(\mathcal{U})$ such that $u = f$ in $X \setminus E$ and $f \in \operatorname{Sob}(\mathcal{U})$ is fixed. That $\mathcal{K}_0$ is a Rellich–Kondrachov cone is guaranteed by Theorem 8.1 in [6]. We remark that neither the Dirichlet nor the obstacle problem have been studied in this setting before.

8. Examples of gradient spaces. II. Other examples

8.1. **Gradient spaces from continuous linear maps.** Let $\widetilde{V}$ and $\widetilde{W}$ be vector spaces, let $V \subseteq \widetilde{V}$ be a reflexive Banach space and let $W \subseteq \widetilde{W}$ be a strictly convex Banach space. Let $D \subseteq V$ be a (norm)-closed linear subspace and let $F : D \to \widetilde{W}$ be a continuous linear map. One defines the graph of F as the relation $R_F \subseteq \widetilde{V} \times \widetilde{W}$ with

$$R_F = \{(u, F(u)) : u \in D\},$$

and one may readily check that R_F is a gradient relation. To show that $\mathcal{U} = (V, \widetilde{V}, W, \widetilde{W}, R_F)$ is a gradient space, one needs to check property (GS4) (note that (GS3) holds since F is linear). Thus, assume that $u, u_i \in V$ and $g, g_i \in W$ are such that $(u_i, g_i) \in R_F$ (i.e. $g_i = F(u_i)$) for $i = 1, 2, \ldots$, and that

$$\|u - u_i\|_V \to 0 \text{ and } \|g - F(u_i)\|_W \to 0.$$

Since the domain of F is assumed to be closed, it follows that $u \in D$, which implies that

$$\|F(u) - F(u_i)\|_W = \|F(u - u_i)\|_W \to 0,$$

as F is continuous. Consequently, $g = F(u)$ which, by definition, implies that $(u, g) \in R_F$.

We shall now provide two more concrete examples of this approach, together with some applications to partial differential equations.

8.2. **More general variational problems and PDEs.** Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and consider $D = V = W^{1,p}(\Omega)$ and $W = L^p(\Omega; \mathbf{R}^n)$, $1 < p < \infty$, e.g. with the norm $\|v\|_W^p = \int_\Omega |v|^p \, dx$. Let $A(x)$ be an $(n \times n)$-matrix with bounded real measurable entries, which is uniformly elliptic in the sense that $|A(x)\xi| \geq \alpha|\xi|$ for some $\alpha > 0$, all $x \in \Omega$ and all $\xi \in \mathbf{R}^n$.

Then $F : W^{1,p}(\Omega) \to L^p(\Omega; \mathbf{R}^n)$, given by $F(u) = A(x)\nabla u$, is a continuous linear mapping and defines a gradient relation R_F as in Section 8.1. The space $\mathrm{Sob}(\mathcal{U})$ obtained in this way is the usual Sobolev space $W^{1,p}(\Omega)$, and it is naturally ordered by the a.e. ordering of functions. Theorems 2.4.1 and 2.5.1 in Ziemer [25], together with the ellipticity of A, imply that the subspace $W_0^{1,p}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, is a regular Rellich–Kondrachov cone (and thus a Poincaré set) also with respect to the gradient relation R_F (cf. Section 7.1). Hence, for every $f \in W^{1,p}(\Omega)$, Theorem 3.2 and Proposition 3.4 provide us with a unique solution of the Dirichlet problem

$$\min \int_\Omega |A(x)\nabla u(x)|^p \, dx$$

among all $u \in W^{1,p}(\Omega)$ with $u - f \in W_0^{1,p}(\Omega)$. This minimizer is a weak solution of the elliptic equation

$$\operatorname{div}(|A(x)\nabla u|^{p-2} A(x)^T A(x)\nabla u) = 0.$$

In particular, if A is the identity matrix, then this is the classical p-Laplace equation $\Delta_p u = 0$.

Similarly, Theorem 5.3 makes it possible to minimize the Rayleigh quotient

$$\min_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |A(x)\nabla u|^p \, dx}{\|u\|_{L^p(\Omega)}^p}.$$

In this setting, there are other possible choices for F. For $V = D$ as above and $W = L^p(\Omega) \times L^p(\Omega; \mathbf{R}^n)$, equipped with the norm

$$\|(v, \xi)\|_W = \left(\int_\Omega (|v|^p + |\xi|^p) \, dx \right)^{1/p},$$

consider

$$F(u)(x) = (\Lambda(u), \partial_1 u(x), \ldots, \partial_n u(x)) \in \mathbf{R} \times \mathbf{R}^n, \tag{8.1}$$

where $\Lambda : W^{1,p}(\Omega) \to L^p(\Omega)$ is any bounded linear operator. The obtained space $\mathrm{Sob}(\mathcal{U})$ is still $W^{1,p}(\Omega)$ and its subspace $W_0^{1,p}(\Omega)$ is a regular Rellich–Kondrachov cone (and thus a Poincaré set). The Dirichlet problem then corresponds to the variational problem

$$\min \int_\Omega (|\Lambda(u)|^p + |\nabla u|^p)\, dx,$$

among all $u \in W^{1,p}(\Omega)$ with $u - f \in W_0^{1,p}(\Omega)$. Some natural choices are $\Lambda(u) = u$ and $\Lambda(u) = u\chi_E$, where χ_E is the characteristic function of a measurable set $E \subset \Omega$. These choices lead to

$$\min \int_\Omega (|u|^p + |\nabla u|^p)\, dx \quad \text{and} \quad \min\biggl(\int_E |u|^p\, dx + \int_\Omega |\nabla u|^p\, dx\biggr).$$

When $\Omega = \mathbf{R}^n$, the obstacle problem, associated with the first functional and the obstacle $\psi = \chi_G$ for an open $G \subset \mathbf{R}^n$, is closely related to the definition of the Sobolev capacity, see e.g. Section 2.35 in Heinonen–Kilpeläinen–Martio [7].

8.3. **Higher-order operators.** Let $V = D = W^{2,2}(\mathbf{R}^n)$ denote the subspace of $L^2(\mathbf{R}^n)$ consisting of functions with first and second order distributional derivatives in $L^2(\mathbf{R}^n)$. Furthermore, let $W = L^2(\mathbf{R}^n)$ and $F(u) = \Delta u$, which clearly is a continuous linear mapping from V to W.

Theorem 4.4.1 in Ziemer [25] implies that for all $u \in W_0^{2,2}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ in $W^{2,2}(\mathbf{R}^n)$, the $W^{2,2}(\mathbf{R}^n)$-norm is equivalent to

$$\sum_{i,j=1}^n \int_\Omega |\partial_{ij} u|^2\, dx.$$

Since $\widehat{F(u)} = -|\xi|^2 \hat{u}$ and $\widehat{\partial_{ij} u} = -\xi_i \xi_j \hat{u}$, where $\hat{u}$ is the Fourier transform of u, Parseval's identity shows that for all $u \in C_0^\infty(\Omega)$, and hence for $u \in W_0^{2,2}(\Omega)$,

$$\begin{aligned}\sum_{i,j=1}^n \int_\Omega |\partial_{ij} u|^2\, dx &= C \sum_{i,j=1}^n \int_{\mathbf{R}^n} |\hat{u}|^2 |\xi_i \xi_j|^2\, d\xi \\ &\le C' \int_{\mathbf{R}^n} |\hat{u}|^2 |\xi|^4\, d\xi = C'' \|\Delta u\|^2_{L^2(\mathbf{R}^n)}.\end{aligned}$$

From this we conclude that $\|u\|_V \le C''' \|\Delta u\|_W$, and it follows that $W_0^{2,2}(\Omega)$ is a Poincaré set with respect to the "gradient" relation

$$R = \{(u, \Delta u) : u \in W^{2,2}(\mathbf{R}^n)\}.$$

Theorem 3.2 and Proposition 3.4 now provide us, for every $f \in W^{2,2}(\mathbf{R}^n)$, with a unique solution of the Dirichlet problem

$$\min \int_\Omega |\Delta u|^2\, dx$$

among all $u \in W^{2,2}(\mathbf{R}^n)$ with $u - f \in W_0^{2,2}(\Omega)$. In terms of partial differential equations, the above minimization problem is equivalent to the biharmonic equation $\Delta^2 u = 0$ with the boundary data $u = f$ and $\nabla u = \nabla f$ on $\partial\Omega$.

Of course, $F(u) = \Delta u$ is not the only choice. One may as well let $F(u)$ be the Hessian matrix of u. Higher-order gradients and operators can also be considered. It is also easy to mix derivatives of different orders in the same way as in (8.1). We will not dwell further upon these extensions here.

8.4. **Gradient spaces from lower semicontinuous sublinear maps.** Let $\widetilde{V}$ be a vector space and let $(\widetilde{W}, \leq)$ be a preordered vector space. A map $F : D \to \widetilde{W}$ is *sublinear* if D is a linear subspace and

(1) $F(u+v) \leq F(u) + F(v)$,
(2) $F(\alpha u) = \alpha F(u)$,

for all $u, v \in D$ and $\alpha > 0$.

Proposition 8.1. *Let $F : D \to \widetilde{W}$ be a sublinear map. Then the set*

$$R_F = \{(u,g) \in \widetilde{V} \times \widetilde{W} : F(u) \leq g\}$$

is a gradient relation on $\widetilde{V} \times \widetilde{W}$.

Proof. (G1) Assume that $(u,g) \in R_F$ and $(u',g') \in R_F$, which implies that $F(u) \leq g$ and $F(u') \leq g'$. As F is sublinear, $F(u+u') \leq F(u) + F(u') \leq g + g'$, which implies that $(u+u', g+g') \in R_F$.

(G2) Assume that $(u,g) \in R_F$ and that $\alpha > 0$. Since f is a sublinear map it holds that $F(\alpha u) = \alpha F(u) \leq \alpha g$ which implies that $(\alpha u, \alpha g) \in R_F$. □

Let $V \subseteq \widetilde{V}$ and $W \subseteq \widetilde{W}$ be reflexive Banach spaces such that W is strictly convex. Furthermore, we assume that D is a closed linear subspace of V, $F(D) \subset W$ and that $F : D \to W$ is sublinear and lower semicontinuous. This immediately implies that properties (GS1)–(GS3) are fulfilled. Let us now show that property (GS4) holds. Assume that $D \ni u_i \to u$ (in V) and $g_i \to g$ (in W) with $(u_i, g_i) \in R_F$ for $i = 1, 2, \ldots$, which implies that $F(u_i) \leq g_i$ for $i = 1, 2, \ldots$. Since D is closed, it is clear that $u \in D$. Then, as F is lower semicontinuous, one obtains

$$F(u) \leq \liminf_{i\to\infty} F(u_i) \leq \liminf_{i\to\infty} g_i = \lim_{i\to\infty} g_i = g,$$

which implies that $(u,g) \in R_F$. Hence, $(V, \widetilde{V}, W, \widetilde{W}, R_F)$ is a gradient space.

Note that the Newtonian spaces and their p-weak upper gradients (as well as the Hajłasz gradients and the gradients given by Poincaré type inequalities) can be seen as a special case of the above construction. We have also seen that in those cases there are plenty of Poincaré sets and regular Rellich–Kondrachov cones. Moreover, in the same spirit as in Section 8.2, the p-weak upper gradients from Section 7.3 can be combined with e.g. u into new sublinear maps, such as the vector-valued map $u \mapsto (u, g_u)$.

8.5. **Matrix algebras.** So far, we have considered examples of gradient spaces based on commutative algebras (of functions), as well as more general, abstract, examples. Let us now illustrate that also noncommutative algebras fit into the framework we have developed.

Let $\mathcal{A}$ be a vector space of (complex or real) matrices of dimension N, and let $A^\dagger$ denote the hermitian transpose of the matrix A. The Frobenius norm

$$\|A\|_{\mathrm{Frob}} = \sqrt{\operatorname{tr} A^\dagger A} \tag{8.2}$$

is strictly convex and, since $\mathcal{A}$ is finite-dimensional, the space $(\mathcal{A}, \|\cdot\|_{\mathrm{Frob}})$ is a uniformly convex Banach space. As in Section 8.4 (with $V = \widetilde{V} = W = \widetilde{W} = \mathcal{A}$), any linear map $F : \mathcal{A} \to \mathcal{A}$ (which is automatically *continuous* since $\mathcal{A}$ is finite-dimensional) induces a gradient relation R_F on $\mathcal{A}$. In particular, one may choose

an inner derivation

$$F(A) = [A, \delta] := A\delta - \delta A$$

for some (fixed) $\delta \in \mathcal{A}$. With such a choice, $\mathcal{U} = (\mathcal{A}, \mathcal{A}, \mathcal{A}, \mathcal{A}, R_F)$ becomes a gradient space.

Moreover, via positive matrices one may introduce a linear ordering. Namely, for matrices A and B one writes $A \geq B$ if the matrix $A - B$ is positive definite or positive semidefinite. With respect to this ordering, $\mathcal{U}$ is an ordered gradient space.

For finite-dimensional matrices one may easily illustrate the fact that $\max(\psi_1, \psi_2)$ is not necessarily comparable to every upper bound of ψ_1 and ψ_2. By setting

$$\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

it is straightforward, but somewhat tedious, to check that the 2×2 identity matrix $\mathbf{1}_2$ minimizes the Frobenius norm among hermitian matrices satisfying $A - \psi_1 \geq 0$ and $A - \psi_2 \geq 0$. Setting

$$A = \begin{pmatrix} 3 & \sqrt{5} \\ \sqrt{5} & 3 \end{pmatrix}$$

one readily checks that $A \geq \psi_1$, $A \geq \psi_2$ and that $A - \mathbf{1}_2$ has the eigenvalues

$$\lambda_+ = 2 + \sqrt{5} > 0$$
$$\lambda_- = 2 - \sqrt{5} < 0.$$

Hence, A is an upper bound for ψ_1 and ψ_2 which is not comparable to $\mathbf{1}_2$.

For self-adjoint algebras of bounded operators on a Hilbert space (which, in some sense, are natural noncommutative examples of our theory), the above situation is more or less generic, since if the hermitian elements of such an algebra form a lattice, then the algebra is commutative [20].

8.6. **Trace ideals.** As concrete infinite-dimensional noncommutative examples of our framework, we consider trace ideals of compact operators (see e.g. Simon [21] for a comprehensive treatment). Let $\mathcal{B}(H)$ denote the algebra of bounded linear operators on a separable Hilbert space H; the scalar product on H will be denoted by $(\cdot, \cdot)$, and the induced norm by $\|\cdot\|_H$. For $A \in \mathcal{B}(H)$ one defines the adjoint operator A^* via $(Ax, y) = (x, A^*y)$, as well as the operator norm, given by

$$\|A\| = \inf_{\|x\|_H = 1} \|Ax\|_H \, .$$

A self-adjoint operator $A \in \mathcal{B}(H)$ (i.e. an operator satisfying $A^* = A$) is called *positive*, and one writes $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. For two operators $A, B \in \mathcal{B}(H)$ one writes $A \geq B$ if $A - B \geq 0$. The operator norm and the adjoint operation makes $\mathcal{B}(H)$ into a C^*-algebra and, in particular, a Banach space. However, this is not the Banach space we want to use for our purposes, since if $\mathcal{B}(H)$ is reflexive as a Banach space with respect to the operator norm, then it is finite-dimensional (see e.g. Takesaki [23, p. 54]). Instead, we shall consider trace ideals together with their associated trace norms.

It is a well-known fact that every nonempty proper ideal $I \subsetneq \mathcal{B}(H)$ is contained in the ideal of compact operators $\mathcal{K}(H)$ ([21, Proposition 2.1]). Furthermore, every compact operator $A \in \mathcal{K}(H)$ has a unique (up to ordering) sequence $\{\mu_i(A)\}_{i \in I}$ of

positive numbers (where I is either a finite or countable index set), called *singular values*, such that for $x \in H$,

$$Ax = \sum_{i \in I} \mu_i(A)(e_i, x)\tilde{e}_i,$$

where both $\{e_i\}_{i \in I}$ and $\{\tilde{e}_i\}_{i \in I}$ are orthonormal sets [21, Proposition 1.4]. In fact, $\{\mu_i(A)\}_{i \in I}$ are the (strictly) positive eigenvalues of the self-adjoint positive operator $|A| = \sqrt{A^*A}$. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. The trace of an operator $A \geq 0$ is defined as the sum

$$\operatorname{tr} A = \sum_{k=1}^{\infty} (Ae_k, e_k) = \sum_{i \in I} \mu_i(A),$$

which may or may not converge (in case it does, the sums are absolutely convergent since every term is nonnegative). For $1 < p < \infty$, we let $L^p(\mathcal{B})$ denote the set of compact operators A for which the sum

$$\|A\|_p := (\operatorname{tr} |A|^p)^{1/p} = \biggl(\sum_{i \in I} \mu_i(A)^p \biggr)^{1/p}$$

converges. (The case $p = 2$ is an infinite-dimensional analogue of the Frobenius norm (8.2).) Recall that $|A|^p$ is defined via the functional calculus for operators (see e.g. Rudin [17, Chapter 12]). The normed space $(L^p(\mathcal{B}), \|\cdot\|_p)$ is a Banach space, and it is a particular example of a *symmetrically normed ideal*, which satisfies

$$\|BA\|_p \leq \|B\| \, \|A\|_p \quad \text{and} \quad \|A\| \leq \|A\|_p \tag{8.3}$$

for $A \in L^p(\mathcal{B})$ and $B \in \mathcal{B}(H)$ ([21, Proposition 2.7]). Moreover, $L^p(\mathcal{B})$ is known to be uniformly convex (Dixmier [4, p. 30] and McCarthy [16, Theorem 2.7]). (Note that the above construction can be generalized to traces on semifinite von Neumann algebras, see e.g. Takesaki [24, Chapter IX] for an introduction to noncommutative integration theory.)

Our aim is to illustrate that one can construct an ordered gradient space with $\widetilde{V} = \widetilde{W} = \mathcal{B}(H)$ and $V = W = L^p(\mathcal{B})$, with respect to the standard ordering on operators as described above. One may introduce a gradient relation on $\mathcal{B}(H) \times \mathcal{B}(H)$ in many different ways, e.g. as in Section 8.1, where one chooses a bounded (and hence continuous) linear map $T : D \to \mathcal{B}(H)$ for a closed subset $D \subseteq L^p(\mathcal{B})$. Note that, in this context, a bounded operator T is such that there exists $C > 0$ with

$$\|T(A)\|_p \leq C \|A\|_p \quad \text{for all } A \in D.$$

Consequently, by defining the gradient relation

$$R_T = \{(A, T(A)) : A \in D\},$$

it follows that

$$\mathcal{U} = (L^p(\mathcal{B}), \mathcal{B}(H), L^p(\mathcal{B}), \mathcal{B}(H), R_T) \tag{8.4}$$

is a gradient space. Furthermore, we shall prove that $\mathcal{U}$ is in fact an ordered gradient space. The result is stated below in Theorem 8.2, but we postpone the proof until the end of this section.

Theorem 8.2. *Let $T : D \to L^p(\mathcal{B})$ be a bounded linear map defined on a closed subset $D \subseteq L^p(\mathcal{B})$ and let $1 < p < \infty$. Then $\mathcal{U} = \big(L^p(\mathcal{B}), \mathcal{B}(H), L^p(\mathcal{B}), \mathcal{B}(H), R_T\big)$ is an ordered gradient space with respect to the standard partial ordering of positive operators.*

Poincaré sets, with respect to the gradient relation R_T, are given by subsets $\Omega \subseteq D$ such that T is bounded from below on Ω, i.e. there exists $c > 0$ such that

$$\left\|T(A)\right\|_p \geq c \left\|A\right\|_p \quad \text{for all } A \in \Omega.$$

Let us illustrate the fact that there are many operators $T : L^p(\mathcal{B}) \to L^p(\mathcal{B})$ which are bounded from below, by the following example. For any $M \in \mathcal{B}(H)$ we let $\widetilde{T}_M$ denote the multiplication operator induced by M, i.e

$$\widetilde{T}_M(A) = MA$$

for $A \in L^p(\mathcal{B})$. (Recall that, since $L^p(\mathcal{B})$ is an ideal in $\mathcal{B}(H)$, the product MA lies in $L^p(\mathcal{B})$.) From (8.3) it follows that that the operator $\widetilde{T}_M$ is bounded

$$\|\widetilde{T}_M(A)\|_p = \left\|MA\right\|_p \leq \left\|M\right\| \left\|A\right\|_p,$$

and we let T_M denote the rescaled operator $T_M = \widetilde{T}_M/2\left\|M\right\|$, giving

$$\left\|T_M(A)\right\|_p \leq \tfrac{1}{2} \left\|A\right\|_p.$$

Finally, we let $T = \mathbf{1} - T_M$ (where $\mathbf{1}$ denotes the identity operator on $L^p(\mathcal{B})$) and deduce that

$$\left\|T(A)\right\|_p = \left\|A - T_M(A)\right\|_p \geq \left\|A\right\|_p - \left\|T_M(A)\right\|_p \geq \tfrac{1}{2} \left\|A\right\|_p.$$

Hence, any subset of $L^p(\mathcal{B})$ is a Poincaré set with respect to the gradient relation defined by T. Further examples are given by Fredholm operators, for which one may find natural Poincaré sets.

Definition 8.3. Let X and Y be Banach spaces and let $F : X \to Y$ be a bounded linear operator. F is a *Fredholm operator* if

(1) $\operatorname{im} F$ is closed,
(2) $\ker F$ is finite-dimensional,
(3) $\operatorname{coker} F = Y/\operatorname{im} F$ is finite-dimensional.

Next, we show that for a Fredholm operator, the complement of the kernel is a Poincaré set.

Proposition 8.4. *Let X and Y be Banach spaces, and let $F : X \to Y$ be a Fredholm operator. Then there exist a closed subspace $V \subseteq X$ and a constant $C > 0$ such that $X = V \oplus \ker F$ and*

$$\left\|v\right\|_X \leq C \left\|F(v)\right\|_Y \quad \textit{for all } v \in V.$$

Proof. It is a standard fact that a finite-dimensional subspace of a normed space is complemented (see e.g. Rudin [17, Lemma 4.21]), i.e. there exists a closed subspace V such that $X = V \oplus \ker F$. Hence, one may consider the operator $\widetilde{F} = F|_V : V \to \operatorname{im} F$ which is a bijective bounded operator between two Banach spaces (since $\operatorname{im} F$ is closed). By the bounded inverse theorem [17, Corollary 2.12 (b)], there exists a bounded inverse $\widetilde{F}^{-1} : \operatorname{im} F \to V$. Thus, for every $u \in \operatorname{im} F$ there exists a constant $C > 0$ such that

$$\|\widetilde{F}^{-1}(u)\|_X \leq C \left\|u\right\|_Y.$$

In particular, one may choose $u = \widetilde{F}(v)$ (for arbitrary $v \in V$), which gives

$$\|v\|_X \le C\|\widetilde{F}(v)\|_Y = C\,\|F(v)\|_Y$$

and proves the second part of the statement. □

Finally, we will prove Theorem 8.2, i.e. that $\mathcal{U}$ (as defined in (8.4)) is an ordered gradient space. The results below are more or less standard, but we choose to repeat them here for two reasons: firstly, statements in the literature are not adapted to our particular setting and, secondly, we want to facilitate for readers who are not so familiar with operator algebras. Let us start by recalling the following lemma, which we state without proof.

Lemma 8.5 (Lemma 2.6 in McCarthy [16])**.** *For $0 \le A, B \in \mathcal{B}(H)$ and $1 \le p < \infty$ it holds that*

$$\operatorname{tr} A^p + \operatorname{tr} B^p \le \operatorname{tr}(A+B)^p.$$

In particular, if $A, B \in L^p(\mathcal{B})$ and $0 \le A \le B$ then

$$\|B\|_p^p - \|A\|_p^p = \operatorname{tr} B^p - \operatorname{tr} A^p \ge \operatorname{tr}(B-A)^p \ge 0,$$

and thus $\|A\|_p \le \|B\|_p$.

Lemma 8.6. *If $\{A_i\}_{i=1}^\infty$ is a sequence of operators in $L^p(\mathcal{B})$, then*

$$\lim_{i\to\infty} \|A_i\|_p = 0 \implies \lim_{i\to\infty} (A_i x, x) = 0 \text{ for all } x \in H.$$

Proof. Since $\|A_i\| \le \|A_i\|_p$, the Cauchy–Schwarz inequality yields

$$|(A_i x, x)| \le \|A_i x\|_H \,\|x\|_H \le \|A_i\|\,\|x\|_H^2 \le \|A_i\|_p \,\|x\|_H^2\,,$$

from which it follows that $|(A_i x, x)| \to 0$ as $i \to \infty$. □

Lemma 8.7. *Let $\{A_i\}_{i=1}^\infty$ be a sequence in $L^p(\mathcal{B})$ such that $A_i \ge B$ for some $B \in \mathcal{B}(H)$. If there exists $A \in L^p(\mathcal{B})$ such that $\lim_{i\to\infty} \|A - A_i\|_p = 0$ then $A \ge B$.*

Proof. Assume that $\|A - A_i\|_p \to 0$ with $A_i \ge B$ for $i = 1, 2, \ldots$, which implies that

$$\lim_{i\to\infty} ((A - A_i)x, x) = 0,$$

by Lemma 8.6. Next, one may write

$$((A-B)x, x) = ((A - A_i)x, x) + ((A_i - B)x, x) \ge ((A - A_i)x, x),$$

since $A_i \ge B$. As $\lim_{i\to\infty}((A - A_i)x, x) = 0$ it follows that $((A-B)x, x) \ge 0$ for all $x \in H$, which is equivalent to $A \ge B$. □

Proof of Theorem 8.2. This now follows directly from Lemmas 8.5 and 8.7. □

8.7. **Upper gradients of operator-valued functions.** In close analogy with the theory of p-weak upper gradients in Section 7.3 one may introduce upper gradients of operator-valued (or, more generally, Banach space valued) functions, see Heinonen–Koskela–Shanmugalingam–Tyson [9], [10]. In particular, we shall consider functions from a metric measure space (X, d, μ) into the space $L^p(\mathcal{B})$, as introduced in Section 8.6.

Thus, let (X, d, μ) be a metric measure space and let $S^{r,p}$ denote the set of (μ-a.e. equivalence classes of) functions $f : X \to L^p(\mathcal{B})$ such that

$$\|f\|_r = \left(\int_X \|f\|_p^r \, d\mu \right)^{1/r} < \infty.$$

Since $L^p(\mathcal{B})$ is uniformly convex, it is for certain values of r and p known that $S^{r,p}$ is uniformly convex (due to the fact that one can prove a Clarkson inequality in $S^{r,p}$); for instance, one may choose $1 < p \leq 2$ and $r = p$ (see Takahashi–Kato [22] for details).

For $q > 1$, following [9], we say that a nonnegative measurable function $g : X \to \mathbf{R}$ is a q-weak upper gradient of $f \in S^{r,p}$ if

$$\|f(\gamma(a)) - f(\gamma(b))\|_r \leq \int_\gamma g \, ds \tag{8.5}$$

for q-almost every rectifiable curve $\gamma : [a, b] \to X$. We define a gradient relation $R \subseteq S^{r,p} \times \widetilde{W}$, where $\widetilde{W}$ denotes the set of ($L^q(X)$-equivalence classes of) functions $X \to \mathbf{R}$, as follows

$$R = \{(u, g) : u \in S^{r,p},\ g \in \widetilde{W} \text{ and } g \text{ is a } q\text{-weak upper gradient of } u\}.$$

(Or more precisely, there is a representative of g which is a q-weak upper gradient of some representative of u.) As for the weak upper gradients in Section 7.3, it is immediate to check that properties (G1) and (G2) are fulfilled, and so R is indeed a gradient relation. Consequently, we consider the gradient space

$$\mathcal{U} = \big(S^{r,p}, \widetilde{V}, L^q(X), \widetilde{W}, R\big),$$

where $\widetilde{V}$ denotes the space of (μ-a.e. equivalence classes) of functions $X \to L^p(\mathcal{B})$. Properties (GS1) and (GS2), of a gradient space, are fulfilled since both $S^{r,p}$ and $L^q(X)$ are uniformly convex. Property (GS3) is satisfied as it follows immediately from (8.5) that if $g \in L^q(X)$ is a q-weak upper gradient of u, then g is also a q-weak upper gradient of $-u$. The fact that (GS4) holds follows (mutatis mutandis) from the corresponding result for upper gradients considered in Section 7.3 (cf. Björn–Björn [1, Proposition 2.3]). Hence, we conclude that $\mathcal{U}$ is a gradient space.

The existence of Poincaré sets is analogous to the case of real-valued functions in Section 7.3. Namely, Heinonen–Koskela–Shanmugalingam–Tyson [9, Theorem 4.3] shows that whether a Poincaré inequality is supported or not, does not depend on the Banach space in which the functions take their values. Therefore, if a metric measure space X supports a Poincaré inequality for real-valued functions, it supports a Poincaré inequality for functions with values in an arbitrary Banach space.

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